

BOUNDARY OCCUPATION AND REALIZABLE AREA LIFTS OF JORDAN CURVES

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ABSTRACT. For a fixed parametrized Jordan curve c , we study uniform limits of the primitives $t \mapsto \int_0^t y_n dx_n$ along smooth Jordan parametrizations $c_n = (x_n, y_n)$ converging uniformly to c . We prove that the difference between any two such limits is determined by a single measurable density against planar area on the curve. When the two coordinates have finite strong p - and q -variation for $p, q > 1$ with $1/p + 1/q = 1$, every density between zero and one is realizable. The resulting space of primitives is a compact convex set whose extreme points correspond to measurable subsets of the boundary. The Asano–Ike criterion gives inscribed rectangles of every prescribed aspect ratio for this critical class. Examples distinguish uniqueness from existence and show that the anisotropic condition extends beyond finite quadratic variation. The question is motivated by the realization arguments in OpenAI’s Navier–Stokes manuscript; the proofs here use planar separation, conformal approximation, and winding estimates.

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1. INTRODUCTION

A smooth simple closed curve determines more than its enclosed area: each subarc, closed by a straight chord, also determines a signed area. For a continuous curve, one can try to recover these quantities by smooth approximation. The approximating curves should remain simple: their enclosed regions are part of the geometry one wishes to preserve. Does this procedure give a unique limit? If several limits are possible, what distinguishes them?

This question arose from reading OpenAI's manuscript *Finite time blowup for Navier–Stokes* [7]. In its construction for the smoothly forced three-dimensional equations, prescribed averages must be realized by a velocity field whose other quadratic quantities remain compatible with them. A correction with zero auxiliary mean can still change a quadratic flux, so the flux must be recomputed after modifying the velocity. Appendix C and Section 8.7 address this realization step and the recomputation required after modifying the field. We ask an analogous geometric question: which area data can arise from embedded approximations of one fixed curve? The planar results below are independent of the PDE construction.

Our answer identifies where the possible ambiguity lies. The interior of the curve contributes a fixed amount of area. All differences between limits are carried by planar area on the curve itself. A single measurable density on this boundary determines the difference on every subarc. Every density between zero and one occurs when the coordinate variations are finite at conjugate exponents. This includes finite strong quadratic variation.

1.1. Realizable primitives. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the parameter circle. Throughout the paper, $c = (x, y) : \mathbb{T} \rightarrow \mathbb{R}^2$ is a positively oriented Jordan parametrization: a continuous injective map traversing its image counterclockwise. Write

$$C = c(\mathbb{T}), \quad D = \text{the bounded component of } \mathbb{R}^2 \setminus C,$$

and let m denote planar Lebesgue measure. We use the same letter c for the periodic extension to $\mathbb{R}$. An interval such as $[s, t]$ always refers to this fixed parametrization.

For a smooth curve, the integral of the one-form $\lambda = y \, dx$ gives a convenient signed-area primitive. Green's formula fixes our sign convention: $\int_C \lambda = -m(D)$ for a positively oriented smooth Jordan curve. On an open subarc, the integral differs from its value along the endpoint chord by the negative signed area of the resulting closed path. We study the following limits of these primitives.

Definition 1.1 (Realizable primitive). A function $f \in C([0, 1]; \mathbb{R})$ belongs to $\mathcal{L}(c)$ if there are smooth regular Jordan parametrizations $c_n = (x_n, y_n) : \mathbb{T} \rightarrow \mathbb{R}^2$ such that

$$c_n \longrightarrow c \quad \text{uniformly}, \quad f_n(t) := \int_0^t y_n(u) x'_n(u) \, du \longrightarrow f(t) \quad \text{uniformly on } [0, 1]. \quad (1.1)$$

Here regular means that the derivative is everywhere nonzero. We call f a *realizable primitive* of c , and (c, f) a *realizable lift*.

The convergence in (1.1) concerns all subarcs at once, since their primitive increments are $f_n(t) - f_n(s)$. Every $f \in \mathcal{L}(c)$ satisfies $f(0) = 0$, but its period increment $f(1)$ is usually nonzero. On the covering line the primitive is

$$\tilde{f}(k+t) = kf(1) + f(t), \quad k \in \mathbb{Z}, \quad 0 \leq t < 1. \quad (1.2)$$

Uniform convergence on $[0, 1]$ therefore gives locally uniform convergence on the covering line.

For example, the unit circle $c(t) = (\cos 2\pi t, \sin 2\pi t)$ has the primitive

$$f(t) = -\pi t + \frac{1}{4} \sin 4\pi t.$$

Its period increment is $-\pi$, the negative of the enclosed area. The first theorem below implies that every convergent sequence of primitives obtained by smooth Jordan approximation of this circle has this same limit.

1.2. Boundary occupation. Suppose $c_n \rightarrow c$ uniformly, and let D_n be the bounded Jordan domain enclosed by c_n . Every point of D eventually belongs to D_n , while every point outside $\bar{D}$ eventually lies outside D_n . Thus a weak limit of the area measures of D_n can differ from the area measure of D only on C .

For a measurable set A , write $\mathbf{1}_A$ for its indicator function; the measure $\mathbf{1}_A m$ assigns area $m(A \cap U)$ to a measurable set U . Every weak limit of the measures $\mathbf{1}_{D_n} m$ has the form $(\mathbf{1}_D + \theta \mathbf{1}_C) m$, where $0 \leq \theta \leq 1$ almost everywhere on C . We call θ a *boundary occupation density*. It describes the portion of boundary area included in the approximating domains in the weak limit. Weak convergence here means convergence of integrals against every continuous compactly supported test function. The notation $L^\infty(C, m)$ refers to essentially bounded measurable functions on C , identified when they agree m -almost everywhere.

Theorem 1.2 (Boundary occupation rigidity). *Each $f \in \mathcal{L}(c)$ determines a unique boundary occupation density $\theta_f \in L^\infty(C, m)$, $0 \leq \theta_f \leq 1$. For every sequence realizing f , its enclosed domains satisfy*

$$\mathbf{1}_{D_n} m \rightharpoonup (\mathbf{1}_D + \theta_f \mathbf{1}_C) m. \quad (1.3)$$

For $f, g \in \mathcal{L}(c)$ and $0 \leq s < t \leq 1$,

$$(f - g)(t) - (f - g)(s) = - \int_{c((s,t))} (\theta_f - \theta_g) dm. \quad (1.4)$$

Consequently $f - g$ has total variation at most $m(C)$, and $\mathcal{L}(c)$ is compact in the uniform topology whenever it is nonempty.

The theorem imposes no variation assumption on c or its approximating curves. Its main point is the subarc formula (1.4): the density obtained from the whole enclosed domain controls the difference on each interval. The values on different subarcs are therefore restrictions of one measure. Endpoints do not affect this formula, since a point has zero planar area. In particular, a zero-area trace has at most one realizable primitive.

1.3. Classification at critical variation. For a continuous path z in a Euclidean space and $r \geq 1$, its *strong r -variation* on $[s, t]$ is

$$V_r(z; [s, t]) = \left(\sup_{s=t_0 < \dots < t_N=t} \sum_{j=1}^N |z(t_j) - z(t_{j-1})|^r \right)^{1/r}. \quad (1.5)$$

The supremum is over all finite partitions; V_1 is the usual total variation. When the interval is omitted, we mean $[0, 1]$. A curve satisfying $|c(t) - c(s)| \leq K|t - s|^{1/2}$ for some constant K has $V_2(c; [s, t])^2 \leq K^2(t - s)$. Thus finite strong quadratic variation includes the critical Hölder exponent $1/2$, as well as all rectifiable parametrizations.

Given a boundary density θ , define its cumulative boundary area by

$$B_\theta(t) = \int_{c((0,t))} \theta dm, \quad 0 \leq t \leq 1.$$

This function is continuous because the area measure on C has no atoms. Equip the set of densities $0 \leq \theta \leq 1$ with the weak-star topology: $\theta_n \rightarrow \theta$ means $\int_C \theta_n \zeta dm \rightarrow \int_C \theta \zeta dm$ for every integrable function ζ on C .

Theorem 1.3 (Classification of realizable primitives). *Suppose $p, q > 1$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1, \quad V_p(x) < \infty, \quad V_q(y) < \infty.$$

There is a unique $f^- \in \mathcal{L}(c)$ with boundary occupation density zero, and

$$\mathcal{L}(c) = \{f^- - B_\theta : \theta \in L^\infty(C, m), \quad 0 \leq \theta \leq 1\}. \quad (1.6)$$

The map $\theta \mapsto f^- - B_\theta$ is an affine homeomorphism from the density interval with its weak-star topology onto $\mathcal{L}(c)$ with its uniform topology. The extreme points of $\mathcal{L}(c)$ are precisely the primitives associated with indicator densities $\theta = \mathbf{1}_A$, where $A \subset C$ is measurable.

Two limiting cases clarify the classification. If $m(C) = 0$, the space $\mathcal{L}(c)$ consists of a single primitive. If $m(C) > 0$, its freedom is infinite-dimensional. For a measurable set $A \subset C$, the corresponding primitive differs from f^- by $-m(A \cap c((0, t)))$. Choosing the constant density $\theta = 1/2$ gives $f(t) = f^-(t) - m(c((0, t)))/2$. Thus even fractional boundary occupation is realized by a sequence of simple curves. Section 6 gives a positive-area example with a $1/2$ -Hölder parametrization.

The proof has two geometric steps. First, a crosscut—a simple arc with endpoints on C and interior in D —closes a boundary subarc into a Jordan curve. Comparing the enclosed subdomains for two approximations yields (1.4); this is carried out in Section 2. Second, inner and outer conformal approximations provide the densities zero and one. We join selected pieces of these two curves through their intervening annulus. Finer selections then realize any prescribed density. Section 4 proves the quadratic case, including a variation estimate on every parameter interval. Proposition 5.4 extends the scalar classification to conjugate coordinate variation by controlling the actual primitives through the joining and smoothing steps. The winding measures developed in Section 3 record the signed area of each subarc as a measure on the plane.

1.4. A critical condition for inscribed rectangles. The existence of a realizable primitive has a geometric consequence. The criterion of Asano and Ike [1, Theorem 1.1] states that $\mathcal{L}(c) \neq \emptyset$ implies the existence of an inscribed rectangle of every prescribed similarity class. Here the four vertices are distinct points of C ; prescribing the similarity class is equivalent to prescribing the ratio of adjacent side lengths.

We prove existence under a condition that can distinguish the two coordinate directions.

Theorem 1.4 (Critical coordinate variation). *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. If*

$$V_p(x; [0, 1]) < \infty, \quad V_q(y; [0, 1]) < \infty, \quad (1.7)$$

then $\mathcal{L}(c) \neq \emptyset$. Consequently C contains a nondegenerate rectangle of every prescribed similarity class. The conclusions also hold when (1.7) is satisfied by two independent real linear functionals applied to c .

The case $p = q = 2$ includes every curve of finite strong quadratic variation. The unequal-exponent case also covers curves for which $V_2(c) = \infty$. The area enclosed by a thin rectangle of length ℓ and width w is ℓw , even when its squared diameter is much larger. The proof decomposes an arc into excursions from the line through its endpoints: maximal subarcs whose interiors avoid that line. Each excursion and its straight chord bound a Jordan domain, with area at most the product of the two coordinate oscillations. Conjugate exponents allow these areas to be summed at the equality $1/p + 1/q = 1$.

Section 5 proves the estimate and constructs the required approximating sequence. It also gives a thin elliptical double spiral with finite coordinate variations for $(p, q) = (3/2, 3)$, infinite quadratic variation, and no pair of independent linear coordinates having finite r_1 - and r_2 -variation with $1/r_1 + 1/r_2 > 1$. A circular double spiral in Section 6, by contrast, has no

realizable primitive: its accumulated area diverges as the curve approaches the origin. Existence and uniqueness of a primitive are therefore distinct questions, even for zero-area traces.

1.5. Relation to earlier work. Asano and Ike [1] introduced the continuous-lift criterion used above. Their applications include rectifiable and locally monotone curves. The zero-area uniqueness conclusion in Theorem 1.2 is also stated in their Remark 5.6, following a proof reduced to zero-area traces. For positive-area traces, inner and outer approximations can have different limiting enclosed areas. The boundary-density formula (1.4) describes this dependence on the approximation, including the period increment. Positive-area Jordan traces already contain every prescribed rectangle by the density argument in [1, Remark 5.5].

Young integration [10] treats the strict condition $1/p + 1/q > 1$. Boedihardjo and Geng [2, Theorem 2.2] construct simple polygonal interpolants through points of a Jordan curve; their applications include Green's theorem below quadratic variation. Li and Pan [5] combine this interpolation with variation-controlled smoothing and the Asano–Ike criterion to prove the rectangular peg theorem for curves with $V_r(c) < \infty$, $r < 2$. Sauzedde [9, Theorem 0.2 and Section 7] establishes anisotropic winding and Green formulas under $1/p + 1/q > 1$. Theorem 1.4 reaches the equality case by using simplicity of the approximating arcs.

Area lifts and their concatenation identities are part of rough-path theory [4, Chapter 2]. Winding-number moments also occur in the reconstruction of simple curves from iterated-integral signatures [3]. The spatial measures in Section 3 express these area data as measures on the plane. For a fixed c , they are determined by the signed areas of all chord-closed subarcs. Our classification describes which of these data can arise from embedded approximations of c .

2. RIGIDITY OF REALIZABLE PRIMITIVES

Throughout this section, $c = (x, y) : \mathbb{T} \rightarrow \mathbb{R}^2$ is the fixed positively oriented Jordan parametrization, C is its trace, and D is the bounded component of $\mathbb{R}^2 \setminus C$. We use $[0, 1]$ as one period of the parameter. The only regularity assumption here is continuity of c . Alongside $\lambda = y \, dx$, define the symmetric area one-form

$$\alpha = \frac{1}{2}(x \, dy - y \, dx) = \frac{1}{2} d(xy) - \lambda.$$

Its integral around a positively oriented smooth Jordan loop is the area of the enclosed domain.

The first observation concerns entire domains. A point away from C eventually lies on the same side of every sufficiently close Jordan approximation. Thus two limiting domain measures can differ only on C . We will show that this difference also determines the difference between the primitives on each subarc. For zero-area traces, the resulting uniqueness agrees with [1, Remark 5.6], following the zero-area reduction in the proof of its main theorem.

Let $c_n = (x_n, y_n)$ and $\hat{c}_n = (\hat{x}_n, \hat{y}_n)$ be smooth Jordan parametrizations converging uniformly to c . Suppose their normalized λ -primitives

$$f_n(t) = \int_0^t y_n(u) \dot{x}_n(u) \, du, \quad g_n(t) = \int_0^t \hat{y}_n(u) \dot{\hat{x}}_n(u) \, du$$

converge uniformly to $f, g \in \mathcal{L}(c)$, respectively. A dot denotes differentiation in the parameter. Denote the bounded domains of $c_n, \hat{c}_n$ by D_n, D'_n . Stability of winding number at a fixed point of D makes both orientations positive for all sufficiently large n .

The domains lie in a common compact disk. Their indicator measures have densities between zero and one, so each sequence has a weakly convergent subsequence. Winding-number stability determines the limiting density off C . After taking subsequences, we may therefore write

$$\mathbf{1}_{D_n} m \rightharpoonup (\mathbf{1}_D + \theta \mathbf{1}_C) m, \quad \mathbf{1}_{D'_n} m \rightharpoonup (\mathbf{1}_D + \theta' \mathbf{1}_C) m, \quad 0 \leq \theta, \theta' \leq 1. \quad (2.1)$$

Here $\theta, \theta' \in L^\infty(C, m)$. We fix these limiting densities before considering individual subarcs.

Theorem 2.1 (Boundary-density rigidity). *For the two limits in (2.1), set $\sigma = \theta - \theta'$. Then*

$$f(t) - g(t) - f(s) + g(s) = - \int_{c((s,t))} \sigma \, dm, \quad 0 \leq s < t \leq 1. \quad (2.2)$$

In particular,

$$V_1(f - g; [0, 1]) \leq \int_C |\sigma| \, dm \leq m(C). \quad (2.3)$$

If $m(C) = 0$, then $f = g$.

The theorem has a useful local interpretation. If the two approximations occupy the same fraction of the boundary area along an arc, their limiting primitive increments on that arc agree. In particular, the primitive difference is constant along every subarc whose trace has area zero.

To prove this, we join the endpoints of a subarc by a path through the interior. This path cuts the approximating domain into two Jordan domains, whose areas are bounded by the area of a common disk. The argument is therefore available even when closure by a straight chord produces winding functions with unbounded L^1 norms.

For points $a, b \in \mathbb{R}^2$, $[a, b]$ denotes the oriented straight segment from a to b , and $B(a, r)$ denotes the open Euclidean ball with center a and radius r . We write $*$ for concatenation of paths and $\bar{\kappa}$ for the reversal of a path κ . A *crosscut* of a Jordan domain is a simple arc whose endpoints lie on the boundary and whose remaining points lie in the domain.

Lemma 2.2 (A polygonal crosscut with adjustable endpoints). *Let $0 < s < t < 1$. For arbitrarily small $r > 0$, there are disjoint balls $O_s = B(c(s), r)$, $O_t = B(c(t), r)$, a number $\varepsilon(r) \downarrow 0$, and a simple polygonal chain*

$$\kappa = [e_s, P_s] * \Gamma * [P_t, e_t]$$

such that the middle path Γ is compactly contained in D , the endpoints e_s, e_t lie outside $\bar{D}$, and the two straight end segments lie in O_s, O_t . For every sufficiently close Jordan approximation c_n , a subchain κ_n joins $c_n(s_n)$ to $c_n(t_n)$, has its open part in D_n , and satisfies

$$|s_n - s| < \varepsilon(r), \quad |t_n - t| < \varepsilon(r).$$

Moreover, the c_n -arcs between s_n, s and between t_n, t lie in the corresponding outer balls.

Proof. Choose $\varepsilon = \varepsilon(r)$ tending to zero so that the two parameter windows are disjoint and

$$c([s - \varepsilon, s + \varepsilon]) \subset B(c(s), r/2), \quad c([t - \varepsilon, t + \varepsilon]) \subset B(c(t), r/2).$$

Injectivity and compactness allow smaller balls B_s, B_t , with closures in O_s, O_t , whose closures have positive distance from the images of the complementary parameter windows.

Choose a point of D in each smaller ball and connect them by a simple polygonal path $\Gamma_0 \subset D$. Such a path exists because an open connected planar domain is polygonally connected; loops in a finite polygonal path can be removed. Let b be its first arrival at $\bar{B}_t$, and let $a < b$ be its last visit to $\bar{B}_s$ before that arrival. The path $\Gamma = \Gamma_0|[a, b]$ is compactly contained in D , and its interior avoids both closed balls. Put $P_s = \Gamma_0(a)$ and $P_t = \Gamma_0(b)$. Choose exterior points $e_s \in B_s \setminus \bar{D}$ and $e_t \in B_t \setminus \bar{D}$. The two straight end segments and Γ meet only at their prescribed endpoints, so their concatenation is simple.

For large n , the middle path lies in D_n , whereas e_s, e_t lie outside $\bar{D}_n$. All intersections with c_n occur on the two end segments. On the first segment, traversed towards P_s , choose the last intersection; on the second, traversed from P_t , choose the first. Compactness of the intersection sets ensures these choices exist, including in the presence of tangencies or coincident segments. The open subchain between them has no boundary intersection and contains Γ , hence lies in D_n . The separation of the smaller balls from the complementary parameter windows locates the

two parameters as asserted. Uniform convergence and the preceding half-radius inclusions also locate the short endpoint arcs in the outer balls. $\square$

Proof of Theorem 2.1. We first pass from λ to the symmetric area one-form α . Define

$$H_n(t) = \int_{c_n|[0,t]} \alpha, \quad G_n(t) = \int_{\widehat{c}_n|[0,t]} \alpha.$$

They converge uniformly to continuous functions H, G , since

$$H_n(t) = \frac{1}{2}(x_n(t)y_n(t) - x_n(0)y_n(0)) - f_n(t). \quad (2.4)$$

The limiting endpoint terms are the same for both sequences. Thus $H - G = -(f - g)$.

Comparison domains. Fix $0 < s < t < 1$, and apply Lemma 2.2 with a small radius r . Use the same polygonal chain κ for both approximating sequences, and denote the crosscut parameters for the second sequence by s'_n, t'_n , with crosscut κ'_n . The Jordan crosscut theorem gives subdomains $E_n \subset D_n$ and $E'_n \subset D'_n$ whose positively oriented boundaries are

$$c_n|[s_n, t_n] * \overline{\kappa_n}, \quad \widehat{c}_n|[s'_n, t'_n] * \overline{\kappa'_n}.$$

Green's formula gives

$$m(E_n) = H_n(t_n) - H_n(s_n) - \int_{\kappa_n} \alpha, \quad (2.5)$$

and the corresponding identity for E'_n . The measures $\mathbf{1}_{E_n}m, \mathbf{1}_{E'_n}m$ are dominated by $\mathbf{1}_{D_n}m$ and $\mathbf{1}_{D'_n}m$, respectively, and hence by m . After a further subsequence, write their weak limits as μ, μ' . The limits of the full domain measures remain those in (2.1). Put $\nu = \mu - \mu'$, a finite signed measure. For a measure ρ and a Borel set A , write $\rho|_A$ for its restriction to A .

Identification on the boundary. Let $z = c(v) \notin O_s \cup O_t$, with $s < v < t$. A sufficiently short compact subarc around v has positive distance from the complementary parameter arc and from κ . Hence a sufficiently small connected neighborhood U of z avoids, for all large n , the boundary of the other component of $D_n \setminus \kappa_n$. That boundary consists of the complementary c_n -arc and κ_n . The neighborhood contains a fixed point of $\mathbb{R}^2 \setminus \overline{D}$, which is outside D_n for large n . The other component therefore misses all of U , and $\mathbf{1}_{E_n} = \mathbf{1}_{D_n}$ almost everywhere on U . Consequently $\mu|_U = (\mathbf{1}_D + \theta \mathbf{1}_C)m|_U$. The same argument applies to μ' . If instead $v \notin [s, t]$, the boundary of E_n itself avoids a suitable U ; the fixed exterior point then gives $\mathbf{1}_{E_n} = 0$ on U . A countable cover yields

$$\nu|_{C \setminus (O_s \cup O_t)} = \sigma \mathbf{1}_{c((s,t))}m|_{C \setminus (O_s \cup O_t)}. \quad (2.6)$$

Identification away from the boundary. Replace the first boundary loop by

$$\widetilde{L}_n = c_n|[s, t] * [c_n(t), P_t] * \overline{\Gamma} * [P_s, c_n(s)].$$

The difference between their oriented chains is a sum of closed chains contained in O_s and O_t : the changed parameter arcs and all changed end segments lie in those balls. Their winding numbers therefore agree outside the two balls. These loops converge uniformly to

$$\widetilde{L} = c|[s, t] * [c(t), P_t] * \overline{\Gamma} * [P_s, c(s)],$$

which is also the limit of the corresponding loops for the second sequence. Winding-number stability implies that the two subdomain indicators have the same eventual value at every point outside $C \cup O_s \cup O_t \cup \Gamma$. Their common bound by one permits dominated convergence. The polygonal chain Γ has area zero, and the limit measures are dominated by m , so both limit measures vanish on it. Consequently

$$\nu|_{\mathbb{R}^2 \setminus (C \cup O_s \cup O_t)} = 0. \quad (2.7)$$

The densities of ν and $\sigma \mathbf{1}_{c((s,t))}m$ have absolute value at most one. Equations (2.6)–(2.7) give

$$\left| \nu(\mathbb{R}^2) - \int_{c((s,t))} \sigma \, dm \right| \leq 2m(O_s \cup O_t) \leq 4\pi r^2. \quad (2.8)$$

Removal of the endpoint neighborhoods. For a continuous function h on $[0, 1]$, write

$$\omega_h(\delta) = \sup_{\substack{u,v \in [0,1] \\ |u-v| \leq \delta}} |h(u) - h(v)|$$

for its modulus of continuity. Uniform convergence of the primitives gives

$$\limsup_n |H_n(t_n) - H_n(s_n) - H(t) + H(s)| \leq 2\omega_H(\varepsilon),$$

and similarly for G . The two crosscuts share the same middle path. Their integral difference is supported on the two fixed straight end segments. Let K_α bound the pointwise norm of α on the common compact disk. Each end segment lies in a ball of radius r , so their total length is at most $4r$. Hence

$$\left| \int_{\kappa_n} \alpha - \int_{\kappa'_n} \alpha \right| \leq 4K_\alpha r.$$

Subtracting the two identities (2.5), passing to their weak limits, and using (2.8), we obtain

$$\left| H(t) - G(t) - H(s) + G(s) - \int_{c((s,t))} \sigma \, dm \right| \leq 2\omega_H(\varepsilon) + 2\omega_G(\varepsilon) + 4K_\alpha r + 4\pi r^2.$$

Let $r \downarrow 0$, with $\varepsilon(r) \downarrow 0$. This proves (2.2) for interior endpoints, with the sign following from (2.4).

The densities θ, θ' are the fixed limits in (2.1), independent of the interval and the crosscut. Thus the identity holds for every interior pair s, t . The measure $A \mapsto \int_{c(A)} \sigma \, dm$ on the parameter circle has no atoms: each parameter value maps to a single point of zero planar area. Its cumulative function is therefore continuous. Continuity extends (2.2) to the endpoints and the full period. Summing the identity over a parameter partition proves the variation bound. $\square$

Corollary 2.3 (The occupation density of a primitive). *Every $f \in \mathcal{L}(c)$ determines a unique density $\theta_f \in L^\infty(C, m)$, $0 \leq \theta_f \leq 1$. For every sequence realizing f , the domain measures converge along the full sequence to $(\mathbf{1}_D + \theta_f \mathbf{1}_C)m$.*

Proof. Consider two subsequential limits of domain measures, obtained from the same sequence or from two sequences realizing f . By Theorem 2.1, the difference of their boundary densities has integral zero on every parameter arc. After composition with c , the corresponding signed measure on the parameter circle vanishes on every interval, hence is zero. All subsequential limits are therefore equal. Weak compactness gives convergence of the full sequence. $\square$

Corollary 2.4 (Compactness of the realization space). *If $\mathcal{L}(c) \neq \emptyset$, it is compact in the uniform norm. The map $f \mapsto \theta_f$ is a homeomorphism from $\mathcal{L}(c)$ onto*

$$\Theta(c) = \{\theta_f : f \in \mathcal{L}(c)\},$$

equipped with the weak-star topology of $L^\infty(C, m)$. Here weak-star convergence of boundary densities means convergence of their integrals against every function in $L^1(C, m)$.

Proof. Fix $f_0 \in \mathcal{L}(c)$. Define a measure $\hat{\mu}$ on the parameter circle by $\hat{\mu}(A) = m(c(A))$ for Borel sets $A \subset \mathbb{T}$. It is finite and has no atoms. Theorem 2.1 gives, for every $f \in \mathcal{L}(c)$,

$$|(f - f_0)(t) - (f - f_0)(s)| \leq \hat{\mu}((s, t)), \quad \|f - f_0\|_\infty \leq m(C). \quad (2.9)$$

Nonatomicity implies $\sup_{t-s \leq \delta} \widehat{\mu}((s, t)) \rightarrow 0$. Thus the family is uniformly bounded and equicontinuous.

It is closed as well. If $f_j \rightarrow f$ uniformly, with each $f_j \in \mathcal{L}(c)$, choose one smooth Jordan curve from a sequence realizing f_j such that its distance from c and the distance of its normalized primitive from f_j are both less than $1/j$. This diagonal sequence realizes f . Arzelà–Ascoli proves compactness.

The density map is injective by (2.2). Relative to f_0 , that equation identifies $-(f - f_0)$ with the cumulative function of the measure $((\theta_f - \theta_{f_0}) \circ c) \widehat{\mu}$. Interval indicators span a dense subspace of $L^1(\widehat{\mu})$. Hence uniform convergence of these cumulative functions, together with the common bound on the densities, implies weak-star convergence against every L^1 test. Conversely, weak-star convergence gives pointwise convergence of the cumulative functions, and their common modulus (2.9) makes it uniform. The Jordan parameter homeomorphism transports this equivalence to $L^1(C, m)$. $\square$

Compactness of $\mathcal{L}(c)$ concerns the limits that can be obtained; existence of such a limit is a separate question. Section 6 gives Jordan curves for which $\mathcal{L}(c) = \emptyset$.

3. SPATIAL WINDING MEASURES AND A COMMON BOUNDARY DENSITY

The preceding section locates the difference between two primitive limits on the boundary. We now describe the area carried by each limit itself. Closing a subarc by its chord gives a winding number at every point away from the closed path. Its integral is the signed enclosed area. Keeping the winding number as a measure also records where that area lies.

The main estimate bounds the total variation of this measure by the strong quadratic variation of the subarc. It will provide compactness for the approximations constructed in Section 4. A triangle identity then relates the limits on all subarcs.

3.1. Variation on short intervals. We first record why finite strong variation gives uniform control on short parameter intervals. We use V_r as defined in (1.5).

Lemma 3.1. *Let $z : [a, b] \rightarrow \mathbb{R}^d$ be a continuous path with finite strong r -variation, where $r \geq 1$. The function $\omega_z(s, t) = V_r(z; [s, t])^r$, defined for $a \leq s \leq t \leq b$, is jointly continuous and superadditive:*

$$\omega_z(s, u) + \omega_z(u, t) \leq \omega_z(s, t) \quad (a \leq s \leq u \leq t \leq b).$$

Moreover,

$$\lim_{\delta \downarrow 0} \sup_{t-s \leq \delta} V_r(z; [s, t]) = 0. \quad (3.1)$$

These statements also hold on compact intervals of a periodic extension of a continuous closed path of finite strong r -variation.

Proof. Superadditivity follows by joining partitions. To prove continuity at the diagonal, suppose that for some u , arbitrarily short intervals $[u, u+h]$ satisfy $V_r(z; [u, u+h])^r \geq \epsilon > 0$. By continuity of z , choose such an h with

$$\sup_{v, w \in [u, u+h]} |z(v) - z(w)|^r < \epsilon/4,$$

and choose a partition whose increment sum exceeds $3\epsilon/4$. Removing its first increment leaves a sum exceeding $\epsilon/2$ on an interval separated from u . Repeating the construction on a smaller interval between u and that first remaining partition point produces disjoint intervals, each with variation power at least $\epsilon/2$. Superadditivity contradicts finiteness of $V_r(z; [a, b])^r$. The same argument applies to intervals ending at u .

An arbitrary partition has at most one increment crossing u . Splitting that increment and using $|v + w|^r \leq 2^{r-1}(|v|^r + |w|^r)$ shows that variation vanishes on shrinking two-sided neighborhoods as well. A compactness argument gives the uniform statement (3.1).

The variation seminorm is subadditive under concatenation:

$$V_r(z; [s, t]) \leq V_r(z; [s, u]) + V_r(z; [u, t]).$$

Indeed, split each partition increment crossing u , and apply Minkowski's inequality to the two resulting sequences of vectors. Adding or removing a shrinking endpoint interval therefore changes V_r by a quantity tending to zero. This proves joint continuity. A periodic extension has finite variation on each compact interval by a finite number of concatenations, so the same proof applies. $\square$

3.2. Winding measures and the triangle identity. For a continuous closed path ℓ and a point z outside its trace, the winding number $\text{wind}(\ell, z)$ is the degree of the map $(\ell - z)/|\ell - z|$ to the unit circle. This topological definition applies without differentiability of the path.

Let γ be a continuous arc. Closing its restriction to $[s, t]$ by the oriented segment from $\gamma(t)$ to $\gamma(s)$, define

$$W_\gamma(s, t; z) = \text{wind}(\gamma|_{[s, t]} * [\gamma(t), \gamma(s)], z) \quad (3.2)$$

off the trace of this closed path. For a smooth simple arc, the winding function is integrable, as the next lemma shows. Its *winding measure* is the finite signed measure

$$M_{s, t}^\gamma = W_\gamma(s, t; \cdot) m.$$

The same definition applies to piecewise smooth simple arcs and full smooth Jordan loops. For a finite signed measure ρ , we write $\|\rho\|_{\text{TV}} = |\rho|(\mathbb{R}^2)$ for its total variation norm. In particular, $\|h m\|_{\text{TV}} = \int_{\mathbb{R}^2} |h| \, dm$.

For three points $a, b, d \in \mathbb{R}^2$, let $T(a, b, d)$ be the measure whose density is the winding number of $[a, b] * [b, d] * [d, a]$. Thus $T(a, b, d)$ is the triangle's area measure with the sign of $\det(b - a, d - a)$; it is zero when the points are collinear. Its total variation is

$$\|T(a, b, d)\|_{\text{TV}} = \frac{1}{2} |\det(b - a, d - a)|. \quad (3.3)$$

Cancellation of the oppositely oriented chords gives

$$M_{s, t}^\gamma = M_{s, u}^\gamma + M_{u, t}^\gamma + T(\gamma(s), \gamma(u), \gamma(t)), \quad s \leq u \leq t. \quad (3.4)$$

For example, consider a path consisting of two consecutive sides $[a, b] * [b, d]$ of a triangle. Each side has zero winding measure after chord closure; the whole path has winding measure $T(a, b, d)$. This is the additional term in (3.4).

Lemma 3.2 (Quadratic winding estimate). *For a smooth injective arc $\gamma : [s, t] \rightarrow \mathbb{R}^2$ with distinct endpoints,*

$$\|M_{s, t}^\gamma\|_{\text{TV}} \leq \frac{\pi}{4} V_2(\gamma; [s, t])^2. \quad (3.5)$$

The same bound holds for a full smooth Jordan loop and for a degenerate interval, with zero measure in the latter case.

Proof. Let L be the line through the two endpoints. The connected components $I_e = (a_e, b_e)$ of $\{u \in (s, t) : \gamma(u) \notin L\}$ are disjoint intervals. Each corresponding arc leaves L and returns to it; call it an excursion; see Figure 1. Since γ is injective, an excursion and its chord form a Jordan loop. Let D_e be the enclosed domain, $\varepsilon_e \in \{-1, 1\}$ its orientation, and $d_e = \text{diam } \gamma([a_e, b_e])$. The chord and the enclosed domain lie in the convex hull of the excursion. The planar isodiametric inequality therefore gives $m(D_e) \leq \pi d_e^2/4$.

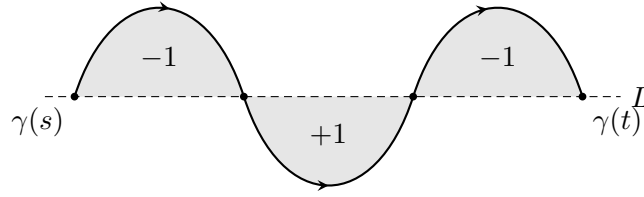


FIGURE 1. An arc traversed from left to right and closed along its endpoint line L . The labels give the winding number in each enclosed region. For these disjoint regions, the total variation of the winding measure is the sum of their areas; its total mass is their signed sum. In the general excursion decomposition, regions may overlap, and the triangle inequality gives the required upper bound.

For a finite excursion family, choose a diameter-realizing pair of times in each closed excursion. Their intervals are disjoint, so all chosen pairs can be included in one ordered parameter partition. Consequently

$$\sum_e m(D_e) \leq \frac{\pi}{4} \sum_e d_e^2 \leq \frac{\pi}{4} V_2(\gamma; [s, t])^2. \quad (3.6)$$

The countable sum follows by finite exhaustion. Replace all but finitely many excursions by their chords on L . These paths converge uniformly to γ : uniform continuity permits only finitely many disjoint excursions of diameter exceeding any fixed positive number. Their winding functions are finite sums of $\varepsilon_e \mathbf{1}_{D_e}$, and (3.6) makes these sums converge in $L^1(m)$. At each point away from the limiting arc and chord, uniform homotopy stability identifies the eventual winding number with that of γ . The arc and chord have area zero, so

$$W_\gamma(s, t; \cdot) = \sum_e \varepsilon_e \mathbf{1}_{D_e} \quad \text{almost everywhere.}$$

The triangle inequality in $L^1(m)$ proves the bound, including when some of the domains overlap or have opposite orientations. For a full Jordan loop, the winding measure is its domain's area measure with the sign of the orientation. The isodiametric inequality and a diameter-realizing pair in a parameter partition give the same estimate. $\square$

For a smooth arc $\gamma = (u, v)$, Green's formula relates its winding measure to its λ -primitive:

$$\int_s^t v \, du = \frac{v(s) + v(t)}{2} (u(t) - u(s)) - M_{s,t}^\gamma(\mathbb{R}^2). \quad (3.7)$$

The first term is the integral of λ along the straight segment from $\gamma(s)$ to $\gamma(t)$. Thus the signed enclosed area measures the difference between integration along the chord and integration along the arc.

3.3. A limit for all subarcs. A bound on each short subarc gives compactness for the entire family of winding measures. The triangle identity is essential here: every measure on $[s, t]$ can be recovered from the two measures on $[0, s]$ and $[0, t]$, together with an endpoint triangle.

Proposition 3.3. *Let c_n be smooth Jordan parametrizations converging uniformly to c . Suppose there are a nonnegative continuous function b on $\Delta = \{(s, t) : 0 \leq s \leq t \leq 1\}$, with $b(t, t) = 0$, and numbers $\eta_n \rightarrow 0$ such that*

$$\|M_{s,t}^{c_n}\|_{\text{TV}} \leq b(s, t) + \eta_n \quad \text{for all } (s, t) \in \Delta. \quad (3.8)$$

There are a subsequence and finite signed Radon measures $M_{s,t}$, $(s,t) \in \Delta$, such that, for every continuous compactly supported function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\sup_{(s,t) \in \Delta} \left| \int_{\mathbb{R}^2} h \, dM_{s,t}^{c_n} - \int_{\mathbb{R}^2} h \, dM_{s,t} \right| \rightarrow 0$$

along that subsequence. The family M is continuous in total variation on Δ and satisfies

$$\|M_{s,t}\|_{\text{TV}} \leq b(s,t), \quad M_{s,s} = 0, \quad (3.9)$$

$$M_{s,t} = M_{s,u} + M_{u,t} + T(c(s), c(u), c(t)), \quad (3.10)$$

$$M_{s,t}|_{\mathbb{R}^2 \setminus c([s,t])} = W_c(s, t; \cdot) m|_{\mathbb{R}^2 \setminus c([s,t])}, \quad (3.11)$$

$$M_{0,1} = (\mathbf{1}_D + \theta \mathbf{1}_C) m, \quad 0 \leq \theta \leq 1. \quad (3.12)$$

The identity (3.10) holds for every $0 \leq s \leq u \leq t \leq 1$; in (3.12), $\theta \in L^\infty(C, m)$. The values assigned to W_c on the closing chord are immaterial in (3.11), because a line segment has planar area zero. The normalized λ -primitives of the same subsequence converge uniformly on $[0, 1]$. Their extensions by (1.2) converge locally uniformly on $\mathbb{R}$.

Proof. All measures have support in a common compact disk K : each winding function vanishes outside the convex hull of its closed path. Their total variations are uniformly bounded. By the Riesz representation theorem, these measures belong to a bounded ball in the dual of $C(K)$, the space of continuous functions on K . That ball is compact and metrizable for weak convergence against continuous functions.

Compactness in the interval variables. Set $F_n(t) = M_{0,t}^{c_n}$. By (3.4),

$$\|F_n(t) - F_n(s)\|_{\text{TV}} \leq b(s,t) + \eta_n + \frac{1}{2} |c_n(s) - c_n(0)| |c_n(t) - c_n(s)|. \quad (3.13)$$

For any prescribed positive error, the term η_n is smaller than that error for all sufficiently large n . Continuity of b and uniform convergence of c_n then give equicontinuity for those indices. Each of the finitely many remaining maps is continuous by Lemma 3.2. This proves equicontinuity of the whole family. The metric-valued Arzelà–Ascoli theorem yields one subsequence converging uniformly in a metric for weak convergence. The formula

$$M_{s,t}^{c_n} = F_n(t) - F_n(s) - T(c_n(0), c_n(s), c_n(t))$$

recovers convergence uniformly over $(s,t) \in \Delta$. Oriented triangle measures are continuous in their vertices in total variation: for a nondegenerate limiting triangle their indicators converge almost everywhere, and for a degenerate one their areas tend to zero. Approximation by a countable dense subset of $C(K)$, together with the uniform total variation bound, gives uniform convergence against each continuous test function. Lower semicontinuity gives (3.9), and the triangle identity passes to (3.10). Passing to the limit in (3.13) proves total variation continuity of $t \mapsto M_{0,t}$. The formula for $M_{s,t}$ then proves this continuity on all of Δ .

Identification away from the arc. Fix $(s,t) \in \Delta$ and a compact ball B disjoint from $c([s,t])$. For large n , the straight homotopy between $c|_{[s,t]}$ and $c_n|_{[s,t]}$ avoids B . Write $a = c(s)$, $d = c(t)$, $a_n = c_n(s)$, and $d_n = c_n(t)$. Segment cancellation gives, almost everywhere on B ,

$$W_{c_n}(s, t) - W_c(s, t) = \text{wind}([a, d] * [d, d_n] * [d_n, a_n] * [a_n, a], \cdot). \quad (3.14)$$

The quadrilateral is the sum of triangles (a, d, d_n) and (a, d_n, a_n) . The sum of their total variation norms is at most a constant, depending only on the common compact disk, times $\|c_n - c\|_\infty$. This also proves local integrability of W_c on B , by comparison with any one sufficiently large n . Consequently the difference in (3.14) converges to zero in $L^1(B)$. A countable cover proves (3.11);

the same argument applies to neighborhoods meeting the closing chord away from $c([s, t])$. For a full loop, winding-number stability off C gives the assertion directly.

The full-period measure and the primitive. Fix $z \in D$. Winding-number stability shows that c_n is positively oriented for all large n . If D_n denotes the bounded domain of c_n , then $M_{0,1}^{c_n} = \mathbf{1}_{D_n} m$, and its weak limit lies between zero and m . Away from C , its density is the winding number of c , so the Radon–Nikodym theorem gives (3.12). Finally, (3.7) gives uniform convergence of the normalized primitives on $[0, 1]$. Their period extensions converge locally uniformly by (1.2). $\square$

The interval bound needed above follows from the estimates in the realization theorem. More generally, suppose $V_2(c; [0, 1]) < \infty$, set $\omega(s, t) = V_2(c; [s, t])^2$, and suppose

$$V_2(c_n; [s, t]) \leq \sqrt{a\omega(s - \delta_n, t + \delta_n)} + e_n$$

for a fixed constant $a \geq 0$ and numbers $\delta_n, e_n \rightarrow 0$. The enlarged intervals are evaluated on the periodic extension of c . Lemmas 3.1 and 3.2 give (3.8) with $b(s, t) = \pi a \omega(s, t)/4$ and a uniform error $\eta_n \rightarrow 0$.

3.4. One boundary density determines every subarc. The possible boundary contribution to $M_{0,1}$ determines the corresponding contribution to every $M_{s,t}$. Indeed, the difference of two families is additive, and its restriction to a parameter interval is supported on the corresponding subarc.

Proposition 3.4 (Measure rigidity). *Suppose M and M' are two total-variation-continuous families satisfying (3.10)–(3.12) for the same c , with $M_{s,s} = M'_{s,s} = 0$ and boundary densities θ, θ' . Then, for every $0 \leq s < t \leq 1$,*

$$M_{s,t} - M'_{s,t} = (\theta - \theta') \mathbf{1}_{c((s,t))} m. \quad (3.15)$$

In particular, the scalar function $t \mapsto M_{0,t}(\mathbb{R}^2)$ determines the measure family uniquely.

Proof. Set $N_{s,t} = M_{s,t} - M'_{s,t}$. The triangle terms cancel, so N is additive on ordered intervals. Equation (3.11) gives $\text{supp } N_{s,t} \subset c([s, t])$. For a proper interval and $s < u < t$, the support of $N_{u,t}$ excludes $c(s)$. Hence

$$|N_{s,t}(\{c(s)\})| \leq \|N_{s,u}\|_{\text{TV}} \rightarrow 0 \quad \text{as } u \downarrow s.$$

The same argument at t removes the other endpoint. The full-period difference has no atoms because it is $(\theta - \theta') \mathbf{1}_C m$. Now split the period at s, t :

$$(\theta - \theta') \mathbf{1}_C m = N_{0,s} + N_{s,t} + N_{t,1}.$$

On $c((s, t))$ the other two terms vanish by support. Outside its closure $N_{s,t}$ vanishes, and its endpoints carry no mass. This proves (3.15).

If the scalar functions agree, the measure $(\theta - \theta') m|_C$ has zero mass on every initial parameter arc. Pulling it back by the Jordan homeomorphism gives a finite nonatomic signed measure whose cumulative function is zero. It vanishes on every interval and hence on every Borel set. Equation (3.15) then gives equality of the families. $\square$

3.5. Relation to iterated integrals. The signed area $A_{s,t} = M_{s,t}(\mathbb{R}^2)$ determines the second-order iterated integral tensor. Writing $c_{s,t} = c(t) - c(s)$ for the path increment, define

$$\mathbb{X}_{s,t} = \frac{1}{2} c_{s,t} \otimes c_{s,t} + A_{s,t} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.16)$$

Here $\otimes$ is the tensor product, identified with the matrix $(v \otimes w)^{ij} = v^i w^j$. For a smooth path this convention gives

$$\mathbb{X}_{s,t}^{ij} = \int_s^t (c^i(r) - c^i(s)) \, dc^j(r), \quad i, j \in \{1, 2\}.$$

Its symmetric part depends only on the endpoints; its antisymmetric part is the signed area. The triangle identity becomes

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + c_{s,u} \otimes c_{u,t}.$$

If $V_2(c; [0, 1]) < \infty$ and $\|M_{s,t}\|_{\text{TV}} \leq K\omega(s, t)$ for a constant K , the pair $(c, \mathbb{X})$ is a weakly geometric 2-rough path in the terminology of [4]. Thus the winding measures give a spatial description of the usual second-order area data. The convergence proved in this section concerns the primitives in the uniform norm and the measures against continuous test functions.

4. REALIZATION OF BOUNDARY OCCUPATION DENSITIES

We now prove that every boundary occupation density between zero and one can be realized when c has finite strong 2-variation. The geometry is easiest to see for the densities zero and one. Curves approaching C from inside enclose domains tending to D ; curves approaching from outside enclose domains tending to $\overline{D}$. The difference is precisely the area carried by C . By joining pieces of an inner curve to pieces of an outer curve, we can choose which parts of that boundary area to include. Finer choices then produce an arbitrary density.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \mapsto x + iy$. Throughout this section, $c : \mathbb{T} \rightarrow \mathbb{C}$ has the orientation and notation fixed in the introduction, and $V_2(c; [0, 1]) < \infty$. We use its periodic extension to $\mathbb{R}$. The variation control

$$\omega(s, t) = V_2(c; [s, t])^2$$

therefore also makes sense for intervals extending beyond $[0, 1]$. By Lemma 3.1, it is continuous and tends to zero uniformly as the interval length tends to zero. This local control will allow all subarc winding measures to converge along the same sequence.

4.1. Conformal approximation in the prescribed parameter.

Lemma 4.1 (Poisson approximation and local variation). *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing homeomorphism satisfying $h(t+1) = h(t) + 1$, and put $b = c \circ h^{-1}$. Let h_n be smooth increasing diffeomorphisms satisfying the same translation identity and converging uniformly to h , and let $0 < \rho_n \uparrow 1$. Write*

$$P_\rho(v) = \frac{1 - \rho^2}{1 - 2\rho \cos(2\pi v) + \rho^2}, \quad \int_{-1/2}^{1/2} P_\rho(v) \, dv = 1,$$

for the periodic Poisson kernel, where $0 < \rho < 1$, and define

$$Q_n(t) = \int_{-1/2}^{1/2} P_{\rho_n}(v) b(h_n(t) - v) \, dv.$$

There are nonnegative numbers $\delta_n, e_n \rightarrow 0$ such that, simultaneously for $0 \leq s \leq t \leq 1$,

$$V_2(Q_n; [s, t]) \leq \sqrt{\omega(s - \delta_n, t + \delta_n)} + e_n. \quad (4.1)$$

Moreover, $Q_n \rightarrow c$ uniformly.

Proof. For $\tau_{n,v}(t) = h^{-1}(h_n(t) - v)$, Minkowski's inequality applied to the vector of increments over any finite partition gives

$$V_2(Q_n; [s, t]) \leq \int_{-1/2}^{1/2} P_{\rho_n}(v) V_2(c; [\tau_{n,v}(s), \tau_{n,v}(t)]) \, dv. \quad (4.2)$$

Each $\tau_{n,v}$ is increasing and has degree one, so an interval of length at most one has image of length at most one. Splitting that image at an integer, if necessary, and using concatenation subadditivity gives

$$V_2(c; [\tau_{n,v}(s), \tau_{n,v}(t)]) \leq L, \quad L = 2V_2(c; [0, 1]).$$

Choose $\eta_n \downarrow 0$ sufficiently slowly that

$$\beta_n = \int_{|v| > \eta_n} P_{\rho_n}(v) \, dv \longrightarrow 0.$$

Uniform continuity of the degree-one inverse h^{-1} and convergence of h_n yield $\delta_n \rightarrow 0$ with

$$|\tau_{n,v}(t) - t| \leq \delta_n \quad (t \in [0, 1], |v| \leq \eta_n).$$

The central part of (4.2) is consequently at most $\sqrt{\omega(s - \delta_n, t + \delta_n)}$, and its remaining part is at most $L\beta_n$. This proves (4.1). The approximate identity property and uniform convergence of h_n give uniform convergence of Q_n . $\square$

The proof gives the same estimate for all real $s \leq t \leq s + 1$: the displacement $\tau_{n,v}(t) - t$ is periodic, and the bound L is uniform over all shifted periods. We will use this extension when reparametrizing the approximating curves.

The homeomorphism h compares the prescribed parameter with the boundary parameter of a conformal map. Poisson averaging takes place in the latter parameter. The enlarged interval in (4.1) records the small displacement when we return to the former. The bound L also covers intervals crossing the chosen base point.

Proposition 4.2 (Inner and outer lifts). *There are smooth regular positively oriented Jordan parametrizations $c_n^\pm \rightarrow c$ uniformly, satisfying*

$$V_2(c_n^\pm; [s, t]) \leq \sqrt{\omega(s - \delta_n, t + \delta_n)} + e_n, \quad \delta_n, e_n \rightarrow 0, \quad (4.3)$$

simultaneously for $0 \leq s \leq t \leq 1$. Their subarc winding measures converge to families $M_{s,t}^\pm$, uniformly in the interval pair against every continuous spatial test function. These families satisfy

$$M_{0,1}^- = \mathbf{1}_D m, \quad M_{0,1}^+ = (\mathbf{1}_D + \mathbf{1}_C) m, \quad (4.4)$$

$$M_{s,t}^+ - M_{s,t}^- = \mathbf{1}_{c((s,t))} m, \quad \|M_{s,t}^\pm\|_{\text{TV}} \leq \frac{\pi}{4} \omega(s, t). \quad (4.5)$$

The corresponding normalized primitives of $\lambda = y \, dx$ converge uniformly to continuous functions $f^\pm$, and

$$f^+(t) - f^-(t) = -m(c((0, t))). \quad (4.6)$$

Proof. The Riemann map $\phi : \{z : |z| < 1\} \rightarrow D$ extends to a homeomorphism of the closures by Carathéodory's theorem; see, for example, [8]. Hence there is an increasing degree-one homeomorphism $h^- : \mathbb{R} \rightarrow \mathbb{R}$, with $h^-(t + 1) = h^-(t) + 1$, such that

$$c(t) = \phi(e^{2\pi i h^-(t)}).$$

Choose smooth increasing degree-one diffeomorphisms $h_n^- \rightarrow h^-$ uniformly. For instance, convolution of the periodic measure dh^- with smooth nonnegative kernels positive near zero

gives strictly positive derivatives, since every open interval has positive dh^- -measure. Integration and adjustment of the additive constant give the required approximation. Set

$$c_n^-(t) = \phi(\rho_n e^{2\pi i h_n^-(t)}), \quad \rho_n \uparrow 1.$$

These parametrizations are smooth and regular, because $\phi' \neq 0$ in the disk and $(h_n^-)' > 0$. The boundary Poisson formula and Lemma 4.1 prove (4.3). Their bounded domains $\phi(\{|z| < \rho_n\})$ increase to D ; thus their period winding measures converge to $\mathbf{1}_D m$.

For the other side, let

$$\psi : \{z : |z| > 1\} \longrightarrow \mathbb{C} \setminus \overline{D}, \quad \psi(z) = az + b_0 + \sum_{k \geq 1} a_{-k} z^{-k}, \quad a \neq 0,$$

be an exterior Riemann map. Its boundary extension gives an increasing degree-one homeomorphism h^+ with $c(t) = \psi(e^{2\pi i h^+(t)})$. For $R > 1$ its Poisson representation is

$$\psi(Re^{2\pi i u}) = \int_{-1/2}^{1/2} P_{1/R}(v) \psi(e^{2\pi i(u-v)}) dv + a(R - R^{-1})e^{2\pi i u}. \quad (4.7)$$

Indeed, the Poisson integral multiplies the positive first Fourier mode by R^{-1} , whereas exterior evaluation multiplies it by R . The negative modes acquire R^{-k} in both expressions. Equivalently, one may apply the disk Poisson formula to $\psi(1/z) - a/z - b_0$, which extends continuously across $z = 0$.

For smooth increasing $h_n^+ \rightarrow h^+$, put

$$c_n^+(t) = \psi(R_n e^{2\pi i h_n^+(t)}), \quad R_n \downarrow 1.$$

The Poisson term in (4.7) satisfies Lemma 4.1. On any interval of length at most one the correction has 1-variation at most

$$2\pi|a|(R_n - R_n^{-1}) \longrightarrow 0.$$

Since $V_2 \leq V_1$, this proves (4.3) also for c_n^+ . Its exterior is $\psi(\{|z| > R_n\})$. The bounded enclosed domains decrease to $\overline{D}$, so the period measures converge to $(\mathbf{1}_D + \mathbf{1}_C)m$. Both sequences converge uniformly to c , and their orientations are positive.

Lemma 3.2 and Proposition 3.3 give subsequential convergence of the subarc families and the bounds in (4.5). Every subsequential inner limit has period density zero on C , and every subsequential outer limit has period density one. Proposition 3.4 therefore gives uniqueness of these limits, convergence of each full sequence, and the difference formula in (4.5).

Recall that $c = (x, y)$ and $\alpha = (x dy - y dx)/2$. Integrating $\alpha = \frac{1}{2} d(xy) - \lambda$ and then adding the closing chord gives

$$f^\pm(t) = \frac{y(0) + y(t)}{2} (x(t) - x(0)) - M_{0,t}^\pm(\mathbb{C}). \quad (4.8)$$

All measures have a common compact support, so their uniform convergence against continuous spatial tests includes their masses. The same identity for $c_n^\pm$ proves uniform convergence of their primitives. Subtraction gives (4.6). $\square$

In particular, the boundary area of a short subarc is controlled by its quadratic variation:

$$m(c((s, t))) \leq \frac{\pi}{2} \omega(s, t). \quad (4.9)$$

This follows by taking total variation in the difference formula (4.5).

4.2. Finite choices of the inner and outer boundaries. Let $E \subset \mathbb{T}$ be a finite union of intervals. We wish to include the boundary area along $c(E)$ and exclude it along $c(\mathbb{T} \setminus E)$. For example, if $E = (a, b)$, we follow an outer approximating curve from near $c(a)$ to near $c(b)$, then an inner approximating curve for the rest of the circuit. Two short bridges join the pieces. For several intervals we repeat this construction in their cyclic order.

The endpoints of E have images of area zero, so their inclusion does not affect the desired density. We take the intervals to be open and assume that their distinct endpoints avoid the base point. We call these endpoints the *switch parameters*.

Lemma 4.3 (Realization of a finite interval choice). *For each such E , there is a sequence c_n of smooth regular positively oriented Jordan parametrizations converging uniformly to c whose subarc winding families converge to*

$$M_{s,t}^E = M_{s,t}^- + \mathbf{1}_{c(E)} \mathbf{1}_{c((s,t))} m. \quad (4.10)$$

The sequence satisfies, simultaneously for all $0 \leq s \leq t \leq 1$,

$$V_2(c_n; [s, t]) \leq \sqrt{3\omega(s - \delta_n, t + \delta_n)} + e_n, \quad \delta_n, e_n \rightarrow 0. \quad (4.11)$$

Proof. The empty and full choices are covered by Proposition 4.2. Otherwise let $0 < t_1 < \dots < t_N < 1$ be the switch parameters; their number N is even.

We first construct disjoint bridges. For each n , choose pairwise disjoint disks about $c(t_j)$, with diameters tending to zero as $n \rightarrow \infty$. By injectivity and compactness, the disks can be chosen so that their intersections with C lie in shrinking parameter neighborhoods of the respective t_j . Each disk contains a point of D and a point of $\mathbb{C} \setminus \overline{D}$. Choose the inner and outer conformal approximations sufficiently close to c that these points lie inside the inner curve and outside the outer curve, respectively. Passing to subsequences and relabeling preserves (4.3).

Within each disk, choose such a pair of points so that the line through them meets both smooth boundaries transversely. This is possible after a small perturbation. The conformal radius circles have real analytic images, so a transverse line has finitely many intersections with them. Follow the segment from its inner endpoint toward its outer endpoint. Let $v_{j,n}$ be its first exit from the outer domain, and let $u_{j,n}$ be its last point in the closure of the inner domain before $v_{j,n}$. Then

$$\kappa_{j,n} = [u_{j,n}, v_{j,n}]$$

joins the two boundaries, and its interior lies in the intervening open annulus. These straight crosscuts are mutually disjoint and have length bounded by the disk diameters. Their endpoints on either boundary occur in the order $t_1, \dots, t_N$: uniform convergence to c and uniform continuity of its inverse force an endpoint near $c(t_j)$ to have parameter near t_j .

Between consecutive crosscuts, follow the outer boundary when the corresponding interval belongs to E , and follow the inner boundary otherwise. At each switch traverse the crosscut. This gives a simple closed curve: the selected boundary arcs are injective and disjoint except at their designated joins, each crosscut interior avoids both boundaries and all other crosscuts, and the prescribed cyclic order connects all pieces into one cycle.

To parametrize the curve, choose disjoint intervals $U_{j,n}$ around the switch parameters t_j , with endpoints tending to t_j . Use each $U_{j,n}$ to traverse $\kappa_{j,n}$ at constant speed. On every remaining interval use the appropriate $c_n^\pm$, with an increasing affine change of parameter matching the endpoints of the selected boundary arc. For each sign, extend these parameter changes across the intervening intervals by increasing affine interpolation. The resulting degree-one homeomorphisms $\sigma_n^\pm$ converge uniformly to the identity. Thus the piecewise smooth curve r_n agrees on its inner and outer pieces with the corresponding curves

$$\gamma_n^\pm = c_n^\pm \circ \sigma_n^\pm.$$

It converges uniformly to c . By (4.3) and monotonicity of the parameter changes, there are $\delta_n, \zeta_n \rightarrow 0$, common to the two signs, such that

$$V_2(\gamma_n^\pm; [s, t]) \leq \sqrt{\omega(s - \delta_n, t + \delta_n)} + \zeta_n. \quad (4.12)$$

The variation estimate has three contributions: increments within inner pieces, increments within outer pieces, and increments crossing between pieces. The first two are controlled by the respective whole curves γ_n^- and γ_n^+ ; the third is compared directly with c . This avoids a factor proportional to the number of pieces in the leading term.

Set $\varepsilon_n = \|r_n - c\|_\infty$, and let $\ell_n \rightarrow 0$ bound the connector lengths. In an arbitrary partition of $[s, t]$, separate increments whose parameter intervals lie in a single outer piece, a single inner piece, or a single connector. All outer increments together have squared sum at most $V_2(\gamma_n^+; [s, t])^2$: their parameter intervals are disjoint, and their endpoints can be included in one partition for γ_n^+ . The same argument applies to all inner increments. On a straight connector the sum of squared increments is at most its length squared, so the total connector contribution is at most $N\ell_n^2$.

Each remaining partition increment crosses a boundary between pieces. There are at most $J = 2N + 2$ such increments. Replacing its endpoint values by those of c at the same parameter values introduces a vector error of size at most $2\varepsilon_n$. The corresponding original increments have squared sum at most $\omega(s, t)$. Minkowski's inequality bounds the contribution of these remaining increments by

$$(\sqrt{\omega(s, t)} + 2\sqrt{J}\varepsilon_n)^2.$$

With $\omega_n = \omega(s - \delta_n, t + \delta_n)$, we have proved

$$V_2(r_n; [s, t])^2 \leq 2(\sqrt{\omega_n} + \zeta_n)^2 + (\sqrt{\omega_n} + 2\sqrt{J}\varepsilon_n)^2 + N\ell_n^2, \quad (4.13)$$

$$V_2(r_n; [s, t]) \leq \sqrt{3\omega_n} + \sqrt{2\zeta_n^2 + 4J\varepsilon_n^2 + N\ell_n^2}. \quad (4.14)$$

The second inequality follows from Minkowski's inequality applied to these bounds. Both estimates hold for every partition of every $[s, t] \subset [0, 1]$.

The finitely many joins can be smoothed in arbitrarily small parameter neighborhoods with an arbitrarily small V_1 error. To see this locally, transversality supplies a linear coordinate that is strictly increasing on the two oriented arcs adjacent to a corner. Smooth the derivative in that coordinate chart and retain the curve outside a smaller neighborhood. The changes tend to zero in V_1 , and the coordinate remains strictly increasing. At a change of positive speed along one boundary arc, the same argument applies with its tangent coordinate. Compact separation of the finitely many other pieces preserves global injectivity. Denote the resulting smooth regular curve by c_n . Its additional V_2 error is at most its V_1 error, so (4.14) proves (4.11). The smoothing can be confined near the switch neighborhoods. Uniform convergence to the positive curve c implies that the index about any fixed point of D is eventually $+1$; hence c_n is positively oriented.

It remains to identify the enclosed boundary area. By Proposition 3.3, a subsequence of the winding families converges to a family M^E , whose period measure has the form $(\mathbf{1}_D + \theta_E \mathbf{1}_C)m$ for some $0 \leq \theta_E \leq 1$. The local bound from (4.11) is a constant multiple of ω , so Proposition 3.4 applies to M^E and both extreme families. If $[s, t]$ is compactly contained in a component of E , its approximating subarc is eventually an unchanged outer boundary piece, with winding measure

$$M_{\sigma_n^+(s), \sigma_n^+(t)}^{c_n^+}.$$

Proposition 4.2, including uniform convergence in both interval endpoints, identifies its limit as $M_{s,t}^+$. Measure rigidity therefore implies $\theta_E = 1$ almost everywhere on $c((s, t))$. Exhausting the components of E by countably many such intervals gives $\theta_E = 1$ on $c(E)$. The inner pieces give

$\theta_E = 0$ on its complement. The finitely many switch points have area zero. This proves (4.10) for every subsequential limit, and hence proves convergence of the full sequence. $\square$

The number N enters (4.14) only through the errors $\sqrt{J} \varepsilon_n$ and $\sqrt{N} \ell_n$. For each finite choice of intervals these errors can be made arbitrarily small by choosing the disks and the conformal approximation sufficiently fine. The next construction chooses these errors after fixing the number of intervals.

4.3. Measurable densities and the realization space. An intermediate density, such as $\theta \equiv \frac{1}{2}$, is obtained by choosing outer arcs on progressively finer intervals that carry half of the boundary area. Their selection is governed by boundary area in the prescribed parameter. Equal parameter lengths may carry very different amounts of area.

More precisely, let $\hat{\mu}$ be the pullback of $m|_C$ to $\mathbb{T}$: for a Borel set $A \subset \mathbb{T}$,

$$\hat{\mu}(A) = m(c(A)).$$

This is a finite atomless measure. For a density θ on C , write $\hat{\theta} = \theta \circ c$. We will approximate $\hat{\theta} \hat{\mu}$ weakly by measures $\mathbf{1}_E \hat{\mu}$ with E a finite union of intervals. Here weak convergence means convergence of integrals against every continuous function on $\mathbb{T}$.

Theorem 4.4 (Realization of arbitrary boundary occupation). *For every $\theta \in L^\infty(C, m)$ with $0 \leq \theta \leq 1$, there is a sequence c_n of smooth regular positively oriented Jordan parametrizations converging uniformly to c whose subarc winding measures converge, uniformly for $(s, t) \in \Delta = \{0 \leq s \leq t \leq 1\}$ against each continuous spatial test, to*

$$M_{s,t}^\theta = M_{s,t}^- + \theta \mathbf{1}_{c((s,t))} m. \quad (4.15)$$

This sequence satisfies (4.11), and

$$\|M_{s,t}^\theta\|_{\text{TV}} \leq \frac{\pi}{2} \omega(s, t). \quad (4.16)$$

Its normalized λ primitives converge uniformly to

$$f^\theta(t) = f^-(t) - \int_{c((0,t))} \theta \, dm. \quad (4.17)$$

Proof. There are finite unions of intervals $E_k \subset \mathbb{T}$ such that

$$\mathbf{1}_{E_k} \hat{\mu} \rightharpoonup \hat{\theta} \hat{\mu}. \quad (4.18)$$

To construct them, partition $\mathbb{T}$ into intervals whose maximal length tends to zero. In each interval I , select an initial subinterval of $\hat{\mu}$ -mass $\int_I \hat{\theta} \, d\hat{\mu}$. Such a subinterval exists because $\hat{\mu}$ has no atoms. Let E_k be the union of the selected subintervals. In each partition interval, the two measures in (4.18) have the same mass. Their difference against a continuous test is therefore bounded by twice the oscillation of that test on the interval, multiplied by its $\hat{\mu}$ -mass. Summing gives convergence in (4.18). Changing E_k in a shrinking neighborhood of the base point ensures that no switch occurs there and preserves (4.18), since $\hat{\mu}$ has no atoms.

For each k , apply Lemma 4.3 to E_k . We choose a single member c_k of that approximating sequence as follows. Make the uniform error, the enlargement δ_k , and the error in (4.14), including smoothing, smaller than $1/k$. In particular, if N_k is the number of switches, require

$$\sqrt{2N_k + 2} \varepsilon_k < 1/k, \quad \sqrt{N_k} \ell_k < 1/k.$$

Fix a closed disk K whose interior contains C . By choosing the approximations sufficiently close to c , we may place all the selected curves and their enclosed domains in K . Fix a countable set of continuous functions dense in $C(K)$ for the uniform norm. Require, in addition, that the period

winding measure of c_k differ from $(\mathbf{1}_D + \mathbf{1}_{c(E_k)})m$ by less than $1/k$ when tested against the first k functions. For each fixed k , Lemma 4.3 allows all these requirements to be met simultaneously.

The resulting smooth Jordan sequence satisfies (4.11). By (4.18), its period measures converge weakly to $(\mathbf{1}_D + \theta \mathbf{1}_C)m$. Subarc compactness and measure rigidity identify every subsequential limit with (4.15). Hence the subarc families converge uniformly in the stated sense.

For a fixed interval, choose a Borel representative of θ taking values in $[0, 1]$, and extend $\theta_{s,t} = \theta \mathbf{1}_{c((s,t))}$ by zero to the plane. Equation (4.5) implies the identity of signed Radon measures

$$M_{s,t}^\theta = (1 - \theta_{s,t})M_{s,t}^- + \theta_{s,t}M_{s,t}^+.$$

Multiplication here is by a bounded measurable function. Taking variations and using $0 \leq \theta_{s,t} \leq 1$ gives

$$\|M_{s,t}^\theta\|_{\text{TV}} \leq \|M_{s,t}^-\|_{\text{TV}} + \|M_{s,t}^+\|_{\text{TV}} \leq \frac{\pi}{2}\omega(s,t).$$

Finally, (4.8) gives (4.17) and uniform convergence of the primitives. $\square$

We can now identify the whole realization space. Equip

$$\{\theta \in L^\infty(C, m) : 0 \leq \theta \leq 1\}$$

with its weak-star topology: $\theta_n \rightarrow \theta$ means that $\int_C u \theta_n \, dm \rightarrow \int_C u \theta \, dm$ for every $u \in L^1(C, m)$. Densities that agree almost everywhere with respect to $m|_C$ represent the same element. We refer to this convex set as the *density cube*.

Theorem 4.5 (Affine geometry of the realization space). *For the realization space $\mathcal{L}(c)$ of Definition 1.1,*

$$\mathcal{L}(c) = \{f^- - B_\theta : \theta \in L^\infty(C, m), 0 \leq \theta \leq 1\}, \quad B_\theta(t) = \int_{c((0,t))} \theta \, dm. \quad (4.19)$$

The correspondence $\theta \mapsto f^- - B_\theta$ is an affine homeomorphism from the weak-star density cube onto $\mathcal{L}(c)$ with the uniform topology. In particular, $\mathcal{L}(c)$ is compact and convex. Its extreme points are precisely

$$f^- - B_{\mathbf{1}_A}, \quad A \subset C \text{ measurable.}$$

Moreover, $\mathcal{L}(c)$ is a singleton if and only if $m(C) = 0$, and every $f \in \mathcal{L}(c)$ satisfies $f^+ \leq f \leq f^-$ on $[0, 1]$.

Proof. Theorem 2.1 applies to every $f \in \mathcal{L}(c)$ and the inner primitive f^- . It supplies a density $0 \leq \theta \leq 1$ and the difference formula $f - f^- = -B_\theta$. Conversely, Theorem 4.4 realizes every such density. This proves (4.19); in particular, it also includes lifts whose defining approximations satisfy no variation bound.

The pullback measure $\hat{\mu}$ is atomless. Thus the functions B_θ , $0 \leq \theta \leq 1$, have the common modulus

$$|B_\theta(t) - B_\theta(s)| \leq \hat{\mu}((s, t)), \quad 0 \leq s \leq t \leq 1.$$

Weak-star convergence of bounded densities gives convergence of their integrals on every interval. The common modulus upgrades this convergence to uniform convergence. Conversely, uniform convergence of the cumulative functions gives convergence of all interval integrals. Interval step functions are dense in $L^1(\mathbb{T}, \hat{\mu})$, and the densities are uniformly bounded; hence it gives weak-star convergence. The same reasoning shows injectivity. Weak-star compactness of the density cube proves compactness of $\mathcal{L}(c)$, as also follows from Corollary 2.4.

The correspondence is affine. An indicator is an extreme point of the density cube, since its values zero and one force both terms of any convex decomposition to agree with it almost everywhere. If a density is strictly between zero and one on a set of positive measure, there are a positive-measure set A and $\epsilon > 0$ such that $\epsilon \leq \theta \leq 1 - \epsilon$ on A . The two distinct

densities $\theta \pm \epsilon \mathbf{1}_A$ express it as a midpoint. This characterizes the extreme points. The singleton criterion follows by comparing the densities zero and one, and the pointwise bounds follow from $0 \leq B_\theta(t) \leq m(c((0, t)))$ and (4.6). $\square$

An extreme point corresponds to a measurable set $A \subset C$: its boundary area is included with density one, and the complementary boundary area with density zero. The primitive determines these choices through its increments on all subarcs. Proposition 3.4 then determines the associated spatial winding measures. Thus one density specifies both the scalar primitive and where its area is located in the plane.

5. A CRITICAL COORDINATE-VARIATION CRITERION

The area enclosed by an arc depends on both coordinate oscillations. A large vertical oscillation can therefore be compatible with a small area if the horizontal oscillation is sufficiently small. This leads to an existence criterion that allows the two coordinates to have different variation exponents, and the curve itself to have infinite quadratic variation. Recall that the strong r -variation of a continuous scalar path z , for $r \geq 1$, is

$$V_r(z; [s, t]) = \sup_{s=t_0 < \dots < t_k=t} \left(\sum_{j=1}^k |z(t_j) - z(t_{j-1})|^r \right)^{1/r}.$$

The strict inequality $1/p + 1/q > 1$ is the usual range of Young integration [10]; anisotropic winding and Green estimates in this range are given by Sauzedde [9, Theorem 0.2 and Section 7]. At equality, the Jordan property supplies an additional constraint: each excursion from the line through its endpoints encloses at most the area of its coordinate bounding rectangle. Hölder's inequality then sums these areas at the conjugate exponents.

Theorem 5.1 (Critical coordinate variation). *Let $c = (x, y) : \mathbb{T} \rightarrow \mathbb{R}^2$ be a positively oriented Jordan parametrization. Suppose*

$$1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad V_p(x; [0, 1]) < \infty, \quad V_q(y; [0, 1]) < \infty. \quad (5.1)$$

There exists a sequence of smooth regular Jordan parametrizations $c_n = (x_n, y_n) \rightarrow c$ uniformly such that

$$f_n(t) = \int_0^t y_n(u) x'_n(u) \, du$$

converges uniformly on $[0, 1]$. The primitives of the periodic extensions converge locally uniformly on $\mathbb{R}$. In particular, $\mathcal{L}(c) \neq \emptyset$, and $c(\mathbb{T})$ contains a nondegenerate Euclidean rectangle of every prescribed similarity class.

The sequence can be chosen so that the winding measures of its chord-closed subarcs, as defined in Section 3, converge uniformly in (s, t) , against every continuous spatial test function, to a family $M_{s,t}$ satisfying

$$\|M_{s,t}\|_{\text{TV}} \leq V_p(x; [s, t]) V_q(y; [s, t]), \quad 0 \leq s \leq t \leq 1. \quad (5.2)$$

The rectangular conclusion follows from the approximation criterion of Asano–Ike [1, Theorem 1.1]. We construct the limiting primitive directly from smooth curves, including at exponents for which Young integration need not apply. The choice $p = q = 2$ recovers existence for every Jordan parametrization of finite strong quadratic variation. The thin spiral constructed below satisfies (5.1) with $p = 3/2$ and $q = 3$, but has infinite quadratic variation.

5.1. The endpoint winding estimate. For a smooth injective arc $\gamma = (u, v) : [s, t] \rightarrow \mathbb{R}^2$ with distinct endpoints, let $W_\gamma(s, t; z)$ be the winding number of the arc followed by the segment from $\gamma(t)$ to $\gamma(s)$. We use planar Lebesgue measure m , and write $\text{osc}_I z = \sup_{a, d \in I} |z(a) - z(d)|$ for the oscillation of a scalar function on an interval I .

Lemma 5.2 (Anisotropic winding bound). *For conjugate exponents $p, q > 1$,*

$$\|W_\gamma(s, t; \cdot)\|_{L^1(m)} \leq V_p(u; [s, t]) V_q(v; [s, t]). \quad (5.3)$$

The same bound holds for a full smooth Jordan loop.

Proof. Let L be the line through the endpoints. The components $I_e = (a_e, b_e)$ of $\{\tau \in [s, t] : \gamma(\tau) \notin L\}$ are disjoint ordered intervals. Each arc $\gamma|_{[a_e, b_e]}$, closed by its segment on L , is Jordan: its open arc is in one open half-plane and its endpoints are distinct. Let D_e be its bounded domain and $\epsilon_e \in \{-1, 1\}$ its orientation. The domain lies in the convex hull of its boundary, and the closing segment does not enlarge either coordinate range. Consequently

$$m(D_e) \leq \text{osc}_{[a_e, b_e]} u \text{osc}_{[a_e, b_e]} v. \quad (5.4)$$

The bounding rectangle has sides parallel to the coordinate axes, even when L is oblique.

For any finite collection of excursions, select two times realizing the oscillation of u on each closed excursion interval. Ordered together, these pairs form part of one partition of $[s, t]$. Deleting duplicate endpoints gives

$$\sum_e (\text{osc}_{[a_e, b_e]} u)^p \leq V_p(u; [s, t])^p.$$

The corresponding argument for v , on the same excursion intervals, gives the analogous q -power bound. Hölder's inequality and monotone passage through finite collections yield

$$\sum_e m(D_e) \leq \sum_e \text{osc}_{[a_e, b_e]} u \text{osc}_{[a_e, b_e]} v \leq V_p(u; [s, t]) V_q(v; [s, t]). \quad (5.5)$$

We also have the spatial identity

$$W_\gamma(s, t; z) = \sum_e \epsilon_e \mathbf{1}_{D_e}(z) \quad \text{for almost every } z. \quad (5.6)$$

Indeed, retain finitely many excursions and replace every other excursion by its straight segment on L . For each fixed positive diameter, uniform continuity permits only finitely many disjoint excursions with image diameter at least that size. Exhausting the excursions therefore gives paths converging uniformly to γ . The winding functions of the modified paths are the finite sums in (5.6); the intervening portions on L have zero winding away from L . By (5.5), these sums converge in $L^1(m)$. Stability of winding number under uniform perturbations identifies their pointwise limit away from the smooth arc and its chord, a set of full planar measure. This proves (5.6), and its L^1 norm proves (5.3). The domains D_e need not be disjoint, and the winding of the chord-closed arc need not be bounded by one.

For a full Jordan loop, the winding is the signed indicator of its bounded domain. That domain lies in the coordinate bounding rectangle, whose area is at most $\text{osc } u \text{osc } v \leq V_p(u) V_q(v)$. $\square$

5.2. Approximation with coordinate variation bounds. Set

$$v_x(s, t) = V_p(x; [s, t]), \quad v_y(s, t) = V_q(y; [s, t]), \quad b(s, t) = v_x(s, t) v_y(s, t).$$

The finite-variation control argument of Lemma 3.1 applies to each coordinate: v_x and v_y are jointly continuous in the endpoints and tend uniformly to zero as $t - s \rightarrow 0$. In particular b is

continuous and vanishes on the diagonal. For an interval $I = [a, d]$, abbreviate $v_x(I) = v_x(a, d)$, and likewise for v_y and b . For ordered disjoint intervals $I_j \subset [s, t]$, Hölder's inequality gives

$$\sum_j b(I_j) \leq \left(\sum_j v_x(I_j)^p \right)^{1/p} \left(\sum_j v_y(I_j)^q \right)^{1/q} \leq v_x(s, t) v_y(s, t). \quad (5.7)$$

Thus b is a continuous superadditive control. The winding estimate will transfer it to all subarcs of the smooth approximations.

Proof of Theorem 5.1. By the interpolation theorem of Boedihardjo–Geng [2, Theorem 2.2], there are partitions P_n of $[0, 1]$, with mesh $\delta_n \rightarrow 0$, such that the piecewise linear curves $c_n^{\text{pl}} = (x_n^{\text{pl}}, y_n^{\text{pl}})$ through the points $c(P_n)$ in parameter order are Jordan. They converge uniformly to c .

Scalar straight interpolation does not increase r -variation between interpolation nodes, for $r \geq 1$. To verify this, take any finite test partition of a scalar polygon. Moving one interior test point inside its interpolation segment makes the sum of its two neighboring increment powers a convex function of that point. Move it to an interpolation endpoint or to a neighboring test point without decreasing the sum, and delete repeated test points. Repeating gives a test partition at interpolation nodes. For arbitrary $s < t$, let s_n^- and t_n^+ be surrounding nodes of P_n , so

$$s - \delta_n \leq s_n^- \leq s, \quad t \leq t_n^+ \leq t + \delta_n.$$

Applied separately to the coordinates of this same polygon, the interpolation bound gives

$$V_p(x_n^{\text{pl}}; [s, t]) \leq v_x(s_n^-, t_n^+), \quad V_q(y_n^{\text{pl}}; [s, t]) \leq v_y(s_n^-, t_n^+).$$

Smooth each fixed periodic Jordan polygon to a regular smooth embedding c_n , and translate to match its base point, with

$$V_1(c_n - c_n^{\text{pl}}; [0, 1]) \leq e_n, \quad e_n \rightarrow 0.$$

This is the polygon smoothing of Li–Pan [5, Lemma 3.1]. Indeed, adjacent nonzero edge velocities of a Jordan polygon cannot be negative multiples. Near each vertex a linear functional positive on both velocities remains positive on their mollified convex combinations. This gives local regularity and injectivity. Compact separation of parameter pairs outside these finitely many neighborhoods gives global injectivity for small smoothing scale. Periodic convolution of the step-function derivative converges in L^1 , which gives the stated V_1 error. Base-point translation does not change that seminorm. Therefore $c_n \rightarrow c$ uniformly and, simultaneously for every interval,

$$\begin{aligned} V_p(x_n; [s, t]) &\leq v_x(s_n^-, t_n^+) + e_n, \\ V_q(y_n; [s, t]) &\leq v_y(s_n^-, t_n^+) + e_n. \end{aligned} \quad (5.8)$$

For the normalized primitive f_n , Green's formula on the chord-closed arc gives

$$f_n(t) - f_n(s) = \frac{y_n(s) + y_n(t)}{2} (x_n(t) - x_n(s)) - \int_{\mathbb{R}^2} W_{c_n}(s, t; z) \, dm(z). \quad (5.9)$$

Taking R with $\sup_n \|y_n\|_\infty \leq R$, Lemma 5.2 and (5.8) imply

$$|f_n(t) - f_n(s)| \leq R|x_n(t) - x_n(s)| + (v_x(s_n^-, t_n^+) + e_n)(v_y(s_n^-, t_n^+) + e_n). \quad (5.10)$$

Continuity of v_x, v_y , uniform convergence of c_n , and $\delta_n, e_n \rightarrow 0$ make the right side uniformly small on short intervals in the tail of the sequence. Each of the finitely many remaining smooth

primitives is uniformly continuous. Hence (f_n) is equicontinuous; the normalization $f_n(0) = 0$ and (5.10) also give a uniform bound. Arzelà–Ascoli yields, after passing to one subsequence,

$$f_n \longrightarrow f \quad \text{uniformly on } [0, 1].$$

For $k \in \mathbb{Z}$ and $0 \leq u < 1$, the primitive of the periodic extension is

$$\tilde{f}_n(k + u) = kf_n(1) + f_n(u).$$

The corresponding expression for f is continuous at the integers, and convergence is uniform on every compact real interval. Rescaling the parameter to the period used in [1, Theorem 1.1] gives its hypotheses. The theorem supplies a rectangle of each prescribed similarity class on C .

We finally establish the spatial assertion using the same sequence. Put $M_{s,t}^n = W_{c_n}(s, t; \cdot)m$. These measures have common compact support and uniformly bounded total variation. If $T(a, b, d)$ denotes oriented triangle measure, winding additivity gives

$$M_{0,t}^n = M_{0,s}^n + M_{s,t}^n + T(c_n(0), c_n(s), c_n(t)).$$

Consequently

$$\|M_{0,t}^n - M_{0,s}^n\|_{\text{TV}} \leq (v_x(s_n^-, t_n^+) + e_n)(v_y(s_n^-, t_n^+) + e_n) + \frac{1}{2}|c_n(s) - c_n(0)||c_n(t) - c_n(s)|. \quad (5.11)$$

This is the compactness argument of Proposition 3.3 with the continuous control b in place of the quadratic-variation bound. Explicitly, the bounded Radon-measure ball on the common compact support is metrizable and weak-star compact. Estimate (5.11) gives equicontinuity, so metric Arzelà–Ascoli extracts one subsequence for which $M_{0,t}^n$ converges uniformly in t in that weak-star metric. Triangle measures depend continuously in total variation on their vertices. The displayed additivity identity thus gives convergence of all $M_{s,t}^n$, uniformly in the interval pair against every continuous test, and preserves the Chen identity. Lower semicontinuity of total variation and continuity of v_x, v_y give (5.2). This further extraction preserves the already established scalar limit. $\square$

5.3. All boundary densities at conjugate exponents. The critical coordinate estimate also gives the scalar classification. For this purpose, the integral along each approximating curve can be followed directly through the finite joining construction. We first record the smoothing fact that preserves this integral.

Lemma 5.3 (Smoothing with convergence of primitives). *Let $r = (u, v) : \mathbb{T} \rightarrow \mathbb{R}^2$ be a Lipschitz Jordan parametrization. Suppose there is a finite set $S \subset (0, 1)$ such that, near every parameter outside S , r agrees with a C^1 map having nonzero derivative there. The hypothesis includes a two-sided regular neighborhood of the base point on the parameter circle. Then*

$$t \longmapsto \int_0^t v \, du \quad \text{belongs to } \mathcal{L}(r).$$

Proof. Around the finitely many points $r(S)$ choose disjoint closed disks, small enough that their intersections with the retained curve lie in disjoint parameter neighborhoods. Near each exceptional parameter, replace the subarc between the first sphere contact before that parameter and the last sphere contact after it by the straight chord in that disk. This removes every intermediate visit to the disk. The contacts can be chosen away from all exceptional parameters. The retained curve lies outside the disk, and the open chord lies inside it. Thus the replacements together give a Jordan curve r_j , with the original parameter used on every retained portion and affine traversal on each chord. Let the disks and their parameter neighborhoods shrink to the exceptional points.

At an entrance, the retained tangent points into the disk or is tangent to its sphere, whereas the chord has a strictly inward radial component. The two oriented velocities are therefore nonzero and are not negative multiples of one another. A linear functional is positive on both. The corresponding statement holds at an exit. Continuity of the retained derivatives gives a neighborhood on which this functional increases at a positive rate along the entire joined curve. Chord interiors and unchanged regular arcs have the same local property. This argument includes tangential sphere contacts.

The curves r_j have a common Lipschitz bound depending only on r and $|S|$. Indeed, the speed of a chord on $[a, b]$ is $|r(b) - r(a)|/(b - a)$, bounded by the Lipschitz constant of r ; splitting at the finitely many contacts bounds every increment. Let U_j be the union of the replaced parameter intervals. Its length tends to zero, and $r'_j = r'$ almost everywhere outside U_j . The common derivative bound consequently gives

$$r_j \longrightarrow r \text{ uniformly, } \|u'_j - u'\|_{L^1(0,1)} \longrightarrow 0.$$

For each fixed r_j , convolve periodically with a nonnegative smooth approximate identity. The local positive linear functionals remain positive on the convolved increments and derivatives when the support is sufficiently small. A finite cover of the parameter circle gives local regularity and injectivity; compact separation of the remaining parameter pairs gives global injectivity. These smooth Jordan approximations converge uniformly, and their coordinate derivatives converge in L^1 .

Both limits preserve the actual primitive. For Lipschitz curves $r = (u, v)$ and $\tilde{r} = (\tilde{u}, \tilde{v})$,

$$\left\| \int_0^\cdot \tilde{v} d\tilde{u} - \int_0^\cdot v du \right\|_\infty \leq \|\tilde{v} - v\|_\infty \|u'\|_{L^1} + \|\tilde{v}\|_\infty \|\tilde{u}' - u'\|_{L^1}. \quad (5.12)$$

Choose the smoothing scale for each j so that its curve and primitive errors are both less than $1/j$. The resulting sequence proves the assertion in the original parameter. $\square$

Proposition 5.4 (Classification under critical coordinate variation). *Under (5.1), there is an inner reference $f^- \in \mathcal{L}(c)$ such that*

$$\mathcal{L}(c) = \{f^- - B_\theta : \theta \in L^\infty(C, m), 0 \leq \theta \leq 1\}.$$

The density is unique modulo $m|_C$ -almost-everywhere equality. The affine homeomorphism, extreme-point description, and order statements of Theorem 4.5 hold for this class.

Proof. The conformal construction of Proposition 4.2 applies to the two coordinates at their respective exponents. For $z = x$ with exponent p , or $z = y$ with exponent q , Minkowski's inequality in the proof of Lemma 4.1 gives

$$V_r(Q_n^z; [s, t]) \leq \int_{-1/2}^{1/2} P_{\rho_n}(w) V_r(z; [\tau_{n,w}(s), \tau_{n,w}(t)]) dw.$$

The central and tail estimates in that proof apply unchanged with the scalar bound $2V_r(z; [0, 1])$. The exterior correction in (4.7) has vanishing 1-variation in each coordinate. Thus the actual inner and outer curves have common coordinate controls with vanishing parameter displacement and error. Lemma 5.2 and the primitive estimate (5.10) make their normalized primitives precompact in the uniform topology.

Choose limits f^- and f^+ on one common subsequence. The inner domains exhaust D and the outer domains decrease to $\overline{D}$, so their boundary occupation densities are zero and one. Theorem 1.2 gives

$$f^+(t) - f^-(t) = -m(c((0, t))), \quad 0 \leq t \leq 1. \quad (5.13)$$

Fix these two sequences and f^- before choosing a boundary density.

First let E be a finite union of separated intervals with endpoints in $(0, 1)$. Use the annular crosscuts and cyclic joining construction of Lemma 4.3. It gives piecewise smooth Jordan curves $r_n \rightarrow c$, traversing the outer sequence on E and the inner sequence on its complement, with parameter changes tending uniformly to the identity. Each bridge has length tending to zero and lies in a common bounded set. Every partial bridge therefore contributes at most a fixed constant times its length to the $y dx$ primitive. On each retained piece the integral is the corresponding inner or outer primitive increment. Their uniform convergence and continuity, including at the moving joining parameters, give

$$\int_0^t (r_n)_2 d(r_n)_1 \longrightarrow f^-(t) - m(c(E \cap (0, t))) \quad \text{uniformly for } 0 \leq t \leq 1.$$

Here (5.13) identifies the contribution of each selected outer interval. The finitely many bridges are nondegenerate; the retained inner and outer branches are regular. Consequently each r_n satisfies Lemma 5.3. Smoothing it with both errors less than $1/n$ proves that this finite-window primitive is realizable.

For an arbitrary $0 \leq \theta \leq 1$, the atomless pullback measure $\hat{\mu}$ admits finite interval choices E_k whose cumulative masses converge uniformly to B_θ , by the construction in Theorem 4.4. Their endpoints can avoid the base point and be separated by gaps of arbitrarily small $\hat{\mu}$ -mass. Choose one smooth realization for each E_k with both errors tending to zero. This single sequence realizes $f^- - B_\theta$.

The converse and density uniqueness follow from Theorem 1.2. The remaining conclusions use only the finite atomless measure, the cumulative-integral correspondence, and this set equality; the proof of Theorem 4.5 therefore applies. $\square$

Corollary 5.5 (Independent linear coordinates). *The conclusions of Theorem 5.1 hold if (5.1) holds for $u = \ell_1 \circ c$ and $v = \ell_2 \circ c$, where $\ell_1, \ell_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ are independent real linear forms. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the invertible map $L(z) = (\ell_1(z), \ell_2(z))$, and put $d = \det L$. The winding measures in the original plane satisfy*

$$\|M_{s,t}\|_{\text{TV}} \leq |d|^{-1} V_p(u; [s, t]) V_q(v; [s, t]).$$

The rectangles lie on $c(\mathbb{T})$ in its original Euclidean geometry.

Proof. For an excursion domain in Lemma 5.2, the change of variables L gives the area bound $|d|^{-1} \text{osc } u \text{osc } v$. The proof by ordered excursions gives the asserted winding bound. Piecewise linear interpolation commutes with L , and the V_1 smoothing error for each coordinate is bounded by the operator norm of its linear form times the vector error. The resulting smooth approximations to c therefore satisfy the required winding estimate in the original plane. In (5.9), the endpoint term still involves x_n, y_n and tends uniformly to zero on short intervals. We obtain a limit of the $y dx$ primitives and apply the Euclidean rectangle criterion to c . $\square$

5.4. A thin double spiral at the critical exponents. The area bound above is sensitive to the shape of an oscillation. In the following example, the vertical semiaxis of the n -th turn is much larger than the horizontal semiaxis. The products of the two semiaxes are summable, whereas the squares of the larger semiaxes are not. This separates the coordinate criterion from finite quadratic variation.

Proposition 5.6. *There is a zero-area Jordan curve satisfying (5.1) with $p = 3/2$, $q = 3$, with the following properties:*

- (i) *it has infinite quadratic variation;*
- (ii) *no nonzero linear projection is monotone on any arc neighborhood of the origin, and every such projection has infinite 1-variation;*

(iii) *no pair of independent linear coordinates has finite r - and s -variation with $1/r + 1/s > 1$. It contains a nondegenerate rectangle of every prescribed similarity class.*

Proof. Set $\vartheta_0 = 4\pi$, $\varphi = \pi/2$, and

$$\begin{aligned} a_1(\vartheta) &= \frac{\vartheta^{-2/3}}{\log \vartheta}, & a_2(\vartheta) &= \frac{\vartheta^{-1/3}}{\log \vartheta}, \\ \gamma_\psi(\vartheta) &= (a_1(\vartheta) \cos(\vartheta + \psi), a_2(\vartheta) \sin(\vartheta + \psi)), \end{aligned} \quad (5.14)$$

for $\vartheta \geq \vartheta_0$ and $\psi \in \{0, \varphi\}$. The functions a_1, a_2 are the horizontal and vertical semiaxes. The powers $2/3$ and $1/3$ give the variation thresholds $3/2$ and 3 ; the logarithmic factors make the variation finite at those thresholds. Traverse γ_0 towards the origin, return along γ_φ , and close the curve by the outer quarter-ellipse. An explicit periodic parametrization is

$$c(t) = \begin{cases} \gamma_0\left(\frac{\vartheta_0}{1-3t}\right), & 0 \leq t < 1/3, \\ (0, 0), & t = 1/3, \\ \gamma_\varphi\left(\frac{\vartheta_0}{3t-1}\right), & 1/3 < t \leq 2/3, \\ (a_1(\vartheta_0) \cos(3\varphi(1-t)), a_2(\vartheta_0) \sin(3\varphi(1-t))), & 2/3 \leq t \leq 1. \end{cases} \quad (5.15)$$

The formulas agree at $2/3$, and $c(0) = c(1)$. Both semiaxes decrease strictly to zero. Ellipse boundaries with different values of ϑ are therefore strictly nested and disjoint. At a common value of ϑ , the two arm points have distinct phases. Thus each arm is injective and the two arms meet only at their common limiting point, the origin. The outer quarter-ellipse meets the arms only at their initial endpoints. This proves the Jordan property. Reversal of the parameter, if needed for positive orientation, preserves all the properties below.

We first prove the required upper bounds for arbitrary partitions. Let d be a positive C^1 function decreasing to zero. For either $z(\vartheta) = d(\vartheta) \cos(\vartheta + \psi)$ or $z(\vartheta) = d(\vartheta) \sin(\vartheta + \psi)$, the total variation on $I_n = [2\pi n, 2\pi(n+1)]$ is at most $5d(2\pi n)$: the amplitude derivative contributes at most $d(2\pi n)$ and the trigonometric derivative contributes at most $4d(2\pi n)$. For a finite test partition, increments contained in one turn have total r -power at most $5^r d(2\pi n)^r$. For increments joining different turns, use

$$|z(v) - z(u)|^r \leq 2^{r-1}(|z(u)|^r + |z(v)|^r).$$

By the order of the partition, each turn contains endpoints of at most two such crossing increments. Assign turn endpoints consistently to one adjacent turn. An increment ending at infinity has zero terminal value and satisfies the same bound. Hence

$$V_r(z; [\vartheta_0, \infty])^r \leq C_r \sum_{n \geq 2} d(2\pi n)^r$$

whenever the series is finite, where C_r depends only on r . The endpoint ∞ denotes the continuous extension with value zero. For the semiaxes in (5.14),

$$\begin{aligned} \sum_{n \geq 2} a_1(2\pi n)^{3/2} &\asymp \sum_{n \geq 2} \frac{1}{n(\log n)^{3/2}} < \infty, \\ \sum_{n \geq 2} a_2(2\pi n)^3 &\asymp \sum_{n \geq 2} \frac{1}{n(\log n)^3} < \infty. \end{aligned} \quad (5.16)$$

Here $\asymp$ means comparison in both directions by positive constants independent of the summation index. Increasing or decreasing changes of parameter preserve variation. Concatenating the two arms and the smooth outer connector therefore gives finite $V_{3/2}(x)$ and finite $V_3(y)$.

On the first arm, successive positive and negative cosine extrema give increments comparable to $a_1(2\pi n)$; successive sine extrema give increments comparable to $a_2(2\pi n)$. The corresponding series diverge below the exponents in (5.16). Thus

$$V_r(x) = \infty \quad (1 \leq r < 3/2), \quad V_s(y) = \infty \quad (1 \leq s < 3). \quad (5.17)$$

In particular $V_2(c) = \infty$.

For a nonzero linear form $\ell(x, y) = ux + vy$, if $v = 0$, (5.17) gives variation threshold $3/2$, attained with finite variation. If $v \neq 0$, at sine extrema on the first arm the x -coordinate is zero. The same alternating-extrema argument proves infinite s -variation for every $s < 3$. Finite 3-variation follows from the coordinate upper bounds and Minkowski's inequality. Two independent linear forms either both have threshold 3, or one has threshold $3/2$ and the other threshold 3. Their possible reciprocal sums are at most $2/3$ and 1, respectively. This proves the third assertion, and also gives infinite 1-variation for every nonzero projection.

Every neighborhood of the origin contains all sufficiently late turns of the first arm. For $v \neq 0$, the projection ℓ alternates sign at successive sine extrema; for $v = 0$, it does so at successive cosine extrema. Hence no nonzero linear projection is monotone on an arc neighborhood of the origin. Finally the trace is the union of the origin, the smooth outer connector, and countably many compact smooth pieces of the arms. It has planar area zero. Theorem 5.1 supplies the rectangles. $\square$

The example lies outside the finite-2-variation class, the locally monotone class, and the strict coordinate-Young range. Since its trace has area zero, the positive-area criterion of [1, Remark 5.5] also does not apply. In particular, it lies outside the finite isotropic r -variation range $r < 2$ of [2, 5].

The distinction between coordinate and isotropic estimates is visible in the area scale

$$a_1(\vartheta)a_2(\vartheta) = \frac{1}{\vartheta(\log \vartheta)^2},$$

which is integrable at infinity, whereas $a_2(\vartheta)^2$ is not. For comparison, circular spiral arms of radius $\vartheta^{-\kappa}$, $0 < \kappa \leq 1/2$, have divergent turn-area integral $\int^\infty \vartheta^{-2\kappa} d\vartheta$. The alternating-extrema argument makes each coordinate's r -variation infinite whenever $\kappa r \leq 1$; therefore such arms cannot satisfy a finite conjugate pair (5.1). Theorem 5.1 establishes existence under the coordinate variation condition, and Proposition 5.4 identifies every realizable primitive throughout this larger class.

6. POSITIVE-AREA TRACES AND FAILURE OF EXISTENCE

For a curve of finite quadratic variation, positive trace area gives distinct realizable primitives. Zero trace area ensures uniqueness whenever a primitive exists. We give an example of each phenomenon: a classical triangle construction produces a positive-area curve of finite quadratic variation, and a circular double spiral has zero trace area and no realizable primitive.

6.1. A positive-area curve of finite strong quadratic variation. In the triangle construction, each triangle is replaced by two smaller triangles that meet at one vertex. A summable loss of area at successive stages leaves a set of positive area in the limit. At the same time, the diameter decreases by approximately $2^{-1/2}$ whenever the parameter interval is halved. These two estimates give positive area and $1/2$ -Hölder continuity together.

Proposition 6.1. *There exists a $1/2$ -Hölder Jordan parametrization whose trace has positive planar area.*

Proof. We use an affine version of Knopp's triangle construction; see [6, Section 3]. Let $\mathcal{T}$ be the closed triangle with vertices

$$a = (0, 0), \quad b = (1/2, 1/2), \quad d = (1, 0), \quad \mathcal{T} = \text{conv}\{a, b, d\}.$$

Thus $m(\mathcal{T}) = 1/4$ and $\text{diam } \mathcal{T} = 1$. For $0 < \epsilon < 1$, put

$$e = ((1 - \epsilon)/2, 0), \quad h = ((1 + \epsilon)/2, 0).$$

Define affine maps L_ϵ, R_ϵ by the ordered vertex images

$$(L_\epsilon(a), L_\epsilon(b), L_\epsilon(d)) = (a, e, b), \quad (R_\epsilon(a), R_\epsilon(b), R_\epsilon(d)) = (b, h, d).$$

Their image triangles intersect only at b , and each has area $(1 - \epsilon)m(\mathcal{T})/2$. Writing DF for the linear part of an affine map F , we have

$$DL_\epsilon = \begin{pmatrix} 1/2 & 1/2 - \epsilon \\ 1/2 & -1/2 \end{pmatrix}, \quad DR_\epsilon = \begin{pmatrix} 1/2 & \epsilon - 1/2 \\ -1/2 & -1/2 \end{pmatrix}. \quad (6.1)$$

At $\epsilon = 0$, each is $2^{-1/2}$ times an orthogonal matrix. Its perturbation has Euclidean operator norm ϵ , so

$$\|DL_\epsilon\|, \|DR_\epsilon\| \leq 2^{-1/2}(1 + \sqrt{2}\epsilon). \quad (6.2)$$

Choose $\epsilon_n = 2^{-n-2}$ for $n \geq 1$. For a binary word $w = w_1 \cdots w_n$, set

$$F_w = F_{\epsilon_1}^{w_1} \circ \cdots \circ F_{\epsilon_n}^{w_n}, \quad F_\epsilon^0 = L_\epsilon, \quad F_\epsilon^1 = R_\epsilon, \quad \mathcal{T}_w = F_w(\mathcal{T}).$$

Here $|w| = n$ denotes the word length. The sets $K_n = \bigcup_{|w|=n} \mathcal{T}_w$ are nested. The finite constant

$$A = \prod_{j \geq 1} (1 + \sqrt{2}\epsilon_j) \leq \exp \left(\sqrt{2} \sum_j \epsilon_j \right)$$

satisfies, by (6.2),

$$\text{diam } \mathcal{T}_w \leq A2^{-n/2}. \quad (6.3)$$

Associate the triangles at depth n , in binary order, with the dyadic parameter intervals of length 2^{-n} . On the interval corresponding to w , let γ_n traverse the segment from $F_w(a)$ to $F_w(d)$ linearly. The first and second children meet at $F_w(b)$; their starting and ending vertices agree with those of the parent. Consequently all later paths agree at existing dyadic endpoints and remain in the same parent triangle. Equation (6.3) makes γ_n uniformly Cauchy. Its limit $\gamma : [0, 1] \rightarrow \mathcal{T}$ joins a to d .

The image is $K = \bigcap_n K_n$. For any point of K , the finitely branching tree of containing triangles has a nested infinite branch. The associated nested parameter intervals have a common parameter, and (6.3) identifies its image with that point. The limit is injective. Siblings meet only at their common traversal endpoint. A starting endpoint is retained only by successive first children, and an ending endpoint only by successive last children. Thus a shared vertex represents precisely the one dyadic parameter joining its neighboring intervals. Two distinct parameters eventually belong to nonadjacent descendant intervals, or are separated by this endpoint rule, and cannot have the same image.

If $2^{-n-1} < t - s \leq 2^{-n}$, the interval $[s, t]$ meets at most two adjacent depth- n cells, apart from their endpoints. Their images share a traversal endpoint, so

$$|\gamma(t) - \gamma(s)| \leq 2A2^{-n/2} \leq 2\sqrt{2}A\sqrt{t - s}. \quad (6.4)$$

The case $n = 0$ follows directly from $\text{diam } \mathcal{T} = 1$. In particular, every finite parameter partition satisfies

$$\sum_i |\gamma(t_i) - \gamma(t_{i-1})|^2 \leq 8A^2.$$

Both determinants in (6.1) have absolute value $(1 - \epsilon)/2$. Distinct same-level triangles have disjoint interiors. Continuity of measure from above therefore gives

$$m(K) = \frac{1}{4} \prod_{j \geq 1} (1 - \epsilon_j) \geq \frac{1}{4} \left(1 - \sum_{j \geq 1} \epsilon_j \right) = \frac{3}{16}. \quad (6.5)$$

Close γ by the lower semicircle of radius $1/2$, centered at $(1/2, 0)$, from d to a . The open semicircular arc lies in $y < 0$, while $\gamma \subset \mathcal{T} \subset \{y \geq 0\}$, so the closed curve is Jordan. Its trace still has area $m(K)$. Allocate half the parameter interval to each piece. The fixed rescaling preserves the $1/2$ -Hölder bound on each piece; across the joining point use $\sqrt{u} + \sqrt{v} \leq \sqrt{2(u+v)}$. This gives a global $1/2$ -Hölder parametrization, with finite strong quadratic variation. Reverse its orientation if necessary. $\square$

The triangle construction is classical. For these affine maps, (6.2)–(6.4) give the critical regularity needed for the classification of $\mathcal{L}(c)$. The inner and outer realizations have period increments that differ by $m(C) \geq 3/16$. More generally, every measurable choice of a boundary occupation density between zero and one gives a realizable primitive of this curve. The existence of prescribed rectangles on positive-area Jordan traces already follows from the Lebesgue density theorem [1, Remark 5.5].

6.2. Local identification on a smooth subarc. Even when a curve is rough elsewhere, a realizable primitive has the usual integral increment on each smooth subarc. The reason is local: closing that subarc by a fixed polygonal path gives a Jordan domain whose boundary has area zero, so its area is stable under uniform approximation.

Lemma 6.2. *Let $f \in \mathcal{L}(c)$. If $[s, t]$ is contained in a regular smooth open subarc of c , then*

$$f(t) - f(s) = \int_{c|[s,t]} \lambda. \quad (6.6)$$

Equivalently, the corresponding continuous α -primitive has its usual integral increment on that subarc.

Proof. Write D for the bounded domain enclosed by C . Near each endpoint choose a short normal segment, from an exterior point into D , meeting C only at that endpoint. Local regularity and compact parameter separation permit these choices. Connect the two interior endpoints by a polygonal path in D . Remove any finite polygonal loops, retaining the two normal end segments. This yields a simple chain meeting C only at $c(s), c(t)$; its part between them is an interior crosscut γ .

Let c_n realize f , and form the crosscuts γ_n by the last and first intersection choices in Lemma 2.2. Every cluster point of an intersection belongs to the intersection of the fixed chain with C . Consequently its endpoint parameters satisfy $s_n \rightarrow s$ and $t_n \rightarrow t$. The integrals on γ_n converge to that on γ , since only endpoints on fixed straight segments change.

Let E_n be the positive Jordan domain bounded by $c_n|[s_n, t_n] * \overline{\gamma_n}$, and let E be the domain bounded by $c|[s, t] * \overline{\gamma}$. Here $*$ denotes concatenation, and the bar reverses the direction of the crosscut. Their boundaries converge uniformly. The limiting boundary consists of a smooth arc and a polygonal crosscut, so it has area zero. Stability of winding number and the bound $0 \leq \mathbf{1}_{E_n} \leq 1$ therefore give $\mathbf{1}_{E_n} \rightarrow \mathbf{1}_E$ in $L^1(m)$. The normalized symmetric-area primitive associated to f is

$$h(t) = \frac{1}{2}(x(t)y(t) - x(0)y(0)) - f(t),$$

and the corresponding primitives h_n for c_n converge uniformly to h . Green's formula and continuity at the moving endpoints now give

$$h(t) - h(s) = m(E) + \int_{\gamma} \alpha = \int_{c[s,t]} \alpha.$$

The identity $\alpha = \frac{1}{2} d(xy) - \lambda$ yields (6.6). $\square$

6.3. A zero-area trace with no realizable primitive. For a circular spiral, each turn contributes area of the order of the squared radius. The radii may tend to zero while the sum of these contributions diverges. A continuous area primitive cannot extend through the limiting point in this case.

Proposition 6.3. *For every $0 < \kappa \leq 1/2$, there is a zero-area Jordan curve c_κ such that $\mathcal{L}(c_\kappa) = \emptyset$.*

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$, fix an angle $0 < \beta < 2\pi$, and consider the two spiral arms

$$\gamma_0(\vartheta) = \vartheta^{-\kappa} e^{i\vartheta}, \quad \gamma_\beta(\vartheta) = \vartheta^{-\kappa} e^{i(\vartheta+\beta)}, \quad 1 \leq \vartheta < \infty. \quad (6.7)$$

Extend both continuously by the origin at $\vartheta = \infty$. Traverse γ_0 towards the origin, return along γ_β , and join their outer endpoints by a simple arc of the unit circle. Each arm is simple because its radius is strictly decreasing. Equality of a point on the two arms would force equal radii and hence equal parameters, contradicting the angle difference $\beta \notin 2\pi\mathbb{Z}$. Their only common point is the origin. The closing circular arc meets them only at their outer endpoints. Compactifying each arm parameter therefore gives a Jordan parametrization. Choose its positive orientation.

The trace has area zero: it is the union of countably many regular smooth compact arcs, one point, and one circular arc. Suppose it had a realizable primitive f , and let h be its continuous symmetric-area primitive, defined as in the preceding proof. Lemma 6.2 determines the increment between any two points on a truncated arm. Along γ_0 , fix $\vartheta_0 \geq 1$ and write t_U for the parameter at which $c_\kappa(t_U) = \gamma_0(U)$. With the inward traversal, the increment is

$$h(t_U) - h(t_{\vartheta_0}) = \frac{1}{2} \int_{\vartheta_0}^U \vartheta^{-2\kappa} d\vartheta. \quad (6.8)$$

If the positive orientation reverses this traversal, the identity follows by reversing the same interval; its divergent magnitude is unchanged. For $\kappa \leq 1/2$, the right-hand side diverges as $U \rightarrow \infty$. On the other hand, t_U tends to the parameter of the origin, where continuity of h requires a finite limit. This contradiction proves the assertion. $\square$

Thus zero trace area gives uniqueness whenever a realization exists, while existence is a separate condition. The absence of a realizable primitive in Proposition 6.3 does not decide the inscribed-rectangle problem for that curve: the lifting hypothesis in [1, Theorem 1.1] is a sufficient condition for inscription.

7. FURTHER QUESTIONS

The boundary occupation density gives a geometric coordinate on the space of realizable primitives. Under conjugate finite coordinate variation, it determines the change in area on every subarc, and every density between zero and one is realized. This explains both the uniqueness for zero-area traces and the infinite-dimensional family of lifts for positive-area traces. The measure formulation also describes where each subarc contributes area. By Proposition 3.4, this spatial description is determined by c and its realizable primitive f .

For a general Jordan parametrization with $\mathcal{L}(c) \neq \emptyset$, recall the set of realizable boundary densities

$$\Theta(c) = \{\theta_f : f \in \mathcal{L}(c)\} \subset \{\theta \in L^\infty(C, m) : 0 \leq \theta \leq 1\}.$$

Corollary 2.4 identifies $\mathcal{L}(c)$ with this weak-star compact set. The realization theorem proves that the inclusion is an equality under conjugate finite coordinate variation. This leads to the following question: whenever $\mathcal{L}(c)$ is nonempty, must every boundary density be realizable? The crosscut argument still supplies the necessary constraint, but extending the annular construction requires control of primitive increments on the joining arcs.

The coordinate estimate controls enclosed area even when quadratic variation is infinite. Proposition 5.4 shows that it also suffices for arbitrary boundary densities. The optimal constants in the simultaneous spatial realization estimates provide a separate quantitative question.

The rectangular peg problem asks for every prescribed aspect ratio on every Jordan curve; its square case goes back to Toeplitz in 1911 [1]. Theorem 1.4 establishes this conclusion for the critical coordinate-variation class. Proposition 6.3 shows that the continuous-lift criterion has a genuine restriction: some Jordan curves have no realizable primitive. Inscribed rectangles on curves outside this criterion require a different existence argument.

8. FORMAL VERIFICATION

The companion Lean development is available at

<https://github.com/veridiscoversLab/boundary-occupation>.

It proves scalar rigidity in the original parameter, critical existence, and the density classification of Proposition 5.4, including its affine homeomorphism, extreme points, and order endpoints. It also proves compactness of every nonempty realization fiber without a variation assumption. All statements concern the canonical boundary-area measure and one sequence of smooth regular Jordan curves whose actual $y \, dx$ primitives converge uniformly. The Euclidean quadratic-variation hypothesis is explicitly identified with its coordinate formulation. Scalar realization in independent linear coordinates is also checked, including the quadratic endpoint correction to the primitive.

The inner and outer approximations, subarc comparison, annular joining, and primitive-preserving smoothing are proved internally. Two classical results remain explicit theorem hypotheses: Jordan-domain conformal uniformization with Carathéodory boundary correspondence, and the Green area formula for piecewise C^1 Jordan curves. Jordan separation and Schoenflies use the proved Lean development of Álvaro Begué.

The repository records exact theorem statements, dependency revisions, source hashes, and verification results. Compilation and kernel replay used Lean 4.32.1 and pinned mathlib imports on a Linux server. The signed spatial winding-measure estimates, examples, and application of the recent Asano–Ike theorem have separate mathematical proofs and cited dependencies; they are outside this formal verification.

NOTATION

The principal notation is collected here for reference. Each object is defined where it first enters the argument.

Notation	Meaning
$\mathbb{T} = \mathbb{R}/\mathbb{Z}$	The parameter circle; paths are extended periodically to $\mathbb{R}$.
$c = (x, y), c_n$	The fixed positively oriented Jordan parametrization and its smooth Jordan approximations.
C, D, m	The trace $c(\mathbb{T})$, its bounded complementary domain, and planar Lebesgue measure.
λ, α	The one-forms $y \, dx$ and $\frac{1}{2}(x \, dy - y \, dx)$. Their integrals around a positive smooth loop are minus and plus its enclosed area, respectively.
$\mathcal{L}(c), f$	The space of realizable primitives and one of its elements, normalized by $f(0) = 0$.
$\tilde{f}$	The extension with period increment $f(1)$, defined by (1.2).
θ_f, B_θ	The boundary occupation density of a lift and the cumulative boundary area $B_\theta(t) = \int_{c((0,t))} \theta \, dm$.
$\hat{\mu}$	Boundary area pulled back to the parameter circle: $\hat{\mu}(A) = m(c(A))$.
$V_r(z; [s, t])$	Strong r -variation. An omitted interval means $[0, 1]$; V_1 is total variation.
$\omega(s, t)$	The quadratic variation control $V_2(c; [s, t])^2$, used when $V_2(c) < \infty$.
$W_\gamma(s, t; z)$	Winding number about z of the subarc $\gamma _{[s,t]}$ closed by its endpoint chord.
$M_{s,t}^\gamma, M_{s,t}$	The winding measure $W_\gamma(s, t; \cdot)m$, and a limit of such measures.
$\ \rho\ _{\text{TV}}$	Total variation norm of a finite signed measure, $ \rho (\mathbb{R}^2)$.
$T(a, b, d)$	Signed area measure of the oriented triangle with successive vertices a, b, d .
f^-, f^+	Inner and outer limiting primitives, with boundary densities zero and one. They satisfy $f^+ \leq f^-$.
ϕ, ψ	Interior and exterior conformal maps in Section 4.
$\Theta(c)$	The set of boundary densities associated with elements of $\mathcal{L}(c)$.

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